Sky Reconstruction from Cylinder Visibilities

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Visibility

This note will consider the reconstruction of the sky from the measured visibilities from a pair of cylinder antenna arrays. It is assumed that the cylinders fixed and are oriented along the meridian. Each cylinder is populated with **N** feeds spaced uniformly along the length. The output voltage of each feed provides an input of a spatial Fourier transform along the cylinder length. The spatial Fourier transform forms **N** beams along the length of the cylinder.

For a pair of cylinders the visibility between cylinders is formed for each beam. As the sky drifts through the cylinder beam, the visibility for beam **k** is:

$$v_k(\varphi) = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{\tilde{A}_k(\theta,\phi)}{\lambda^2} T(\theta,\varphi-\phi) cos(\theta)\, d\theta d\phi \qquad (1)$$

Where $\boldsymbol{\varphi}$ is the time of the day (in units of angle), $\boldsymbol{\lambda}$ is the wavelength and **T** is the power flux of the sky. The cylinder pair Fourier area is defined as

$$\tilde{A}_k(\theta,\phi) = \tilde{\mathrm{a}}_{\mathrm{k,c1}}(\theta,\phi)\left(\tilde{\mathrm{a}}_{\mathrm{k,c2}}(\theta,\phi)\right)^* \qquad (2)$$

where the subscripts **c1**, **c2** indicate cylinder 1 and cylinder 2, respectively. The Fourier root area of a cylinder is defined as

$$\tilde{\mathrm{a}}_{\mathrm{k,c}}(\theta,\phi) = \sum_n \mathrm{a_n}(\theta,\phi) e^{-j\vec{\beta}(\theta,\phi)\cdot\vec{r}_{n,c}} e^{j2\pi k\frac{n}{N}} \qquad (3)$$

Where **n** is the feed number, $\mathbf{r_{n,c}}$ is the global location of the feed and $\boldsymbol{\beta}$ is the incoming wave vector:

$$\vec{\beta}(\theta,\phi) = \frac{2\pi}{\lambda}(sin(\theta)\hat{x} + cos(\theta)sin(\phi)\hat{y}) \qquad (4)$$

It is assumed that the length of the cylinders is in the x direction.

Sky Expansion

Since the sky is periodic in right ascension, it can be expanded in a Fourier series:

$$T(\theta,\phi) = \sum_l \chi_l(\theta)\left(\tilde{T}_{dc_{l,0}} + \sum_m \tilde{T}_{c_{l,m}} cos(m\phi) + \sum_m \tilde{T}_{s_{l,m}} sin(m\phi)\right) \qquad (5)$$

Substituting Equation 5 into Equation 1,

$$\begin{aligned} &Re\{v_k(\varphi)\} \\ &= \sum_l \hat{A}^{(R)}_{dc\ \ k,l,0}\tilde{T}_{l,0} \\ &+ \sum_l \sum_m \hat{A}^{(R)}_{c\ \ k,l,m}\tilde{T}_{c\,l,m}cos(m\varphi) - \sum_l \sum_m \hat{A}^{(R)}_{s\ \ k,l,m}\tilde{T}_{s\,l,m}cos(m\varphi) \\ &+ \sum_l \sum_m \hat{A}^{(R)}_{s\ \ k,l,m}\tilde{T}_{c\,l,m}sin(m\varphi) + \sum_l \sum_m \hat{A}^{(R)}_{c\ \ k,l,m}\tilde{T}_{s\,l,m}sin(m\varphi) \end{aligned} \quad (6)$$

where:

$$\hat{A}^{(R)}_{dc\ \ k,l,0} = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{Re\{\tilde{A}_k(\theta,\phi)\}}{\lambda^2}\chi_l(\theta)cos(\theta)\,d\theta d\phi \quad (7)$$

$$\hat{A}^{(R)}_{c\ \ k,l,m} = \int_{-\pi}^{\pi} cos(m\phi) \int_{-\pi}^{\pi} \frac{Re\{\tilde{A}_k(\theta,\phi)\}}{\lambda^2}\chi_l(\theta)cos(\theta)\,d\theta d\phi \quad (8)$$

$$\hat{A}^{(R)}_{s\ \ k,l,m} = \int_{-\pi}^{\pi} sin(m\phi) \int_{-\pi}^{\pi} \frac{Re\{\tilde{A}_k(\theta,\phi)\}}{\lambda^2}\chi_l(\theta)cos(\theta)\,d\theta d\phi \quad (9)$$

Fourier Transform of Cylinder Visibilities

Now assume that the visibility is measured at N discrete times during the day. The visibility will be periodic with a period of a day so we can expand the visibility into a Fourier series.

$$Re\{v_k(\varphi_n)\} = \tilde{V}^{(R)}_{dc\ \ k,0} + \sum_{m\prime} \tilde{V}^{(R)}_{c\ \ k,m\prime}cos(m'\varphi_n) + \sum_{m\prime} \tilde{V}^{(R)}_{s\ \ k,m\prime}sin(m'\varphi_n) \quad (10)$$

where

$$\tilde{V}^{(R)}_{dc\ \ k,0} = \frac{1}{N}\sum_n Re\{v_k(\varphi_n)\} \quad (11)$$

$$\tilde{V}^{(R)}_{c\ \ k,m\prime} = \frac{2}{N}\sum_n Re\{v_k(\varphi_n)\}\ cos(m'\varphi_n) \quad (12)$$

$$\tilde{V}^{(R)}_{s\ \ k,m\prime} = \frac{2}{N}\sum_n Re\{v_k(\varphi_n)\}\ sin(m'\varphi_n) \quad (13)$$

Substituting Equations 6 into Equations 8, 9, 10,

$$\tilde{V}^{(R)}_{dc\ \ k,0} = \sum_l \hat{A}^{(R)}_{dc\ \ k,l,0}\tilde{T}_{l,0} \quad (14)$$

$$\tilde{V}^{(R)}_{c\ \ k,m} = \sum_l \hat{A}^{(R)}_{c\ \ k,l,m}\tilde{T}_{c\,l,m} - \hat{A}^{(R)}_{s\ \ k,l,m}\tilde{T}_{s\,l,m} \quad (15)$$

$$\tilde{V}_s^{(R)}{}_{k,m} = \sum_l \hat{A}_s^{(R)}{}_{k,l,m} \tilde{T}_{c\,l,m} + \hat{A}_c^{(R)}{}_{k,l,m} \tilde{T}_{s\,l,m} \tag{16}$$

Sky Mode Matrix Equations

Equation 15 and 16 can be combined if we write:

$$\tilde{V}^{(R)}{}_{k,m} = \tilde{V}_c^{(R)}{}_{k,m} - j\tilde{V}_s^{(R)}{}_{k,m} \tag{17}$$

$$\hat{A}^{(R)}{}_{k,l,m} = \hat{A}_c^{(R)}{}_{k,l,m} - j\hat{A}_s^{(R)}{}_{k,l,m} \tag{18}$$

$$\tilde{T}_{l,m} = \tilde{T}_{c\,l,m} - j\tilde{T}_{s\,l,m} \tag{19}$$

Then:

$$\tilde{V}^{(R)}{}_{k,m} = \sum_l \hat{A}^{(R)}{}_{k,l,m} \tilde{T}_{l,m} \tag{20}$$

where:

$$\tilde{V}^{(R)}{}_{k,m} = \frac{2}{N} \sum_n Re\{v_k(\varphi_n)\}\, e^{-jm\varphi_n} \tag{21}$$

$$\hat{A}_c^{(R)}{}_{k,l,m} = \int_{-\pi}^{\pi} e^{-jm\phi} \int_{-\pi}^{\pi} \frac{Re\{\tilde{A}_k(\theta,\phi)\}}{\lambda^2} \chi_l(\theta) cos(\theta)\, d\theta d\phi \tag{22}$$

$$T(\theta,\phi) = \sum_l \chi_l(\theta) \left(\tilde{T}_{dc\,l,0} + Re\left\{ \sum_m \tilde{T}_{c\,l,m} e^{jm\phi} \right\} \right) \tag{23}$$

Discussion on Sign Convention

Since the sky temperature must be a real function, in Equation 5, we expanded the sky temperature in a real Fourier series. Since the choice of which cylinder is first in the definition of the visibility in Equation 2 is arbitrary, the choice of coordinate system for the incoming wave vector in Equation 3 and 4, and the choice of sign for the mode number m in Equation 5 is will change the signs of the imaginary part of the Fourier transform of the visibility. For example, if the **y** location of cylinder 2 is **greater** than cylinder 1, then for an ideal pair of cylinders:

$$\hat{A}^{(R)}{}_{k,l,m} = \hat{A}^{(R)}{}_{k,l,-m} = j\hat{A}^{(I)}{}_{k,l,m} = -j\hat{A}^{(I)}{}_{k,l,-m} \tag{24}$$

However, if the **y** location of cylinder 2 is **less** than cylinder 1, then for an ideal pair of cylinders then:

$$\hat{A}^{(R)}{}_{k,l,m} = \hat{A}^{(R)}{}_{k,l,-m} = -j\hat{A}^{(I)}{}_{k,l,m} = j\hat{A}^{(I)}{}_{k,l,-m} \tag{25}$$

If one is not careful with the choice of sign of the sky expansion mode number m and the choice of definition of which cylinder visibility, the complex Fourier transform can be matrix element $A_{k,l,m}$ can be zero. For these reasons, the choice of only using the Fourier transform of the real part of the visibilities in Equation 10 was decided.