Formulation of Cylinder Visibilities

Dave McGinnis
February 9, 2010

Feed Amplitude

The voltage received at a feed located at coordinate r is:

$$\frac{v(\vec{r})}{\sqrt{2R}} = \iint_{\Omega} f(\Omega)a(\Omega)e^{-j\vec{\beta}(\Omega)\cdot\vec{r}}d\Omega \tag{1}$$

The sky flux amplitude is:

$$|f(\Omega)d\Omega|^2 = \frac{kT_{sky}(\Omega)}{\lambda^2}d\Omega_{\mathrm{pow}} \tag{2}$$

where $d\Omega_{pow}$ is the differential power solid angle area. The incoming wave vector is:

$$\vec{\beta}\big(\Omega(\theta,\phi)\big) = \frac{2\pi}{\lambda}(sin(\theta)\hat{x} + cos(\theta)sin(\phi)\hat{y}) \tag{3}$$

The collecting area of the feed is:

$$A(\Omega) = |a(\Omega)|^2 \tag{4}$$

The noise power generated by the feed amplifier is:

$$P_z = |p_z|^2 \tag{5}$$

If the sky is pixelized into **q** pixels then, the signal amplitude at feed **n** of cylinder **m** is:

$$p_{n,m} = p_{z_{n,m}} + \sum_q \Delta\Omega_{\mathrm{q}}\mathrm{f}_{\mathrm{q}}\mathrm{a}_{\mathrm{q,n,m}}e^{-j\vec{\beta}_q\cdot\vec{r}_{n,m}} \tag{6}$$

Cylinder Amplitude

A spatial Fourier transform will be taken of the cylinder feed voltage.

$$\tilde{p}_{k,m} = \tilde{p}_{z_{k,m}} + \tilde{\psi}_{k,m} \tag{7}$$

where:

$$\tilde{p}_{z_{k,m}} = \sum_n p_{z_{n,m}}e^{j2\pi k\frac{n}{N}} \tag{8}$$

$$\tilde{\psi}_{k,m} = \sum_q \Delta\Omega_{\mathrm{q}}\mathrm{f}_{\mathrm{q}}\tilde{\mathrm{a}}_{\mathrm{q,k,m}} \tag{9}$$

$$\tilde{\mathrm{a}}_{\mathrm{q,k,m}} = \sum_n \mathrm{a}_{\mathrm{q,n,m}}e^{-j\vec{\beta}_q\cdot\vec{r}_{n,m}}e^{j2\pi k\frac{n}{N}} \tag{10}$$

Cylinder Visibility

The visibility between cylinder **m** and cylinder **m’** for beam k is:

$$v_{k,m,m\prime} = \tilde{p}_{k,m}\left(\tilde{p}_{k,m\prime}\right)^{*} \tag{11}$$

Because the cross terms in noise power vanish, the time average of the visibility is:

$$\langle v_{k,m,m\prime}\rangle = \langle \tilde{p}_{k,m}\left(\tilde{p}_{k,m\prime}\right)^{*}\rangle = \langle \tilde{\psi}_{k,m}\left(\tilde{\psi}_{k,m\prime}\right)^{*}\rangle \tag{12}$$

$$\langle v_{k,m,m\prime}\rangle = \sum_{q} \langle \left|\Delta\Omega_{\mathrm{q}} \mathrm{f}_{\mathrm{q}}\right|^{2}\rangle\, \tilde{\mathrm{a}}_{\mathrm{q,k,m}} \left(\tilde{\mathrm{a}}_{\mathrm{q,k,m}\prime}\right)^{*} \tag{13}$$

For Gaussian noise distributions for signal amplitudes,

$$\langle v_{k,m,m\prime}\rangle = \sum_{q} \Delta\Omega_{\mathrm{pow}_{\mathrm{q}}} \frac{\langle kT_{sky}(\Omega_{\mathrm{q}})\rangle}{\lambda^{2}} \tilde{\mathrm{a}}_{\mathrm{q,k,m}} \left(\tilde{\mathrm{a}}_{\mathrm{q,k,m}\prime}\right)^{*} \tag{14}$$

Variation of cylinder visibility

The variation in the real part of the visibility is given as:

$$\langle\left(Re\left(v_{k,m,m\prime} - \langle v_{k,m,m\prime}\rangle\right)\right)^{2}\rangle \tag{15}$$

which can be expanded to:

$$\langle\left(Re\left(v_{k,m,m\prime} - \langle v_{k,m,m\prime}\rangle\right)\right)^{2}\rangle = \langle\left(Re\{v_{k,m,m\prime}\}\right)^{2}\rangle - \left(Re\{\langle v_{k,m,m\prime}\rangle\}\right)^{2} \tag{16}$$

It can be shown that:

$$\left(Re\{v_{k,m,m\prime}\}\right)^{2} = \frac{1}{2} Re\left\{\left(\tilde{p}_{k,m}\right)^{2}\left(\left(\tilde{p}_{k,m\prime}\right)^{*}\right)^{2}\right\} + \frac{1}{2}\left|\tilde{p}_{k,m}\right|^{2}\left|\tilde{p}_{k,m\prime}\right|^{2} \tag{17}$$

The average value of the first term is:

$$\langle Re\left\{\left(\tilde{p}_{k,m}\right)^{2}\left(\left(\tilde{p}_{k,m\prime}\right)^{*}\right)^{2}\right\}\rangle = Re\left\{\langle\left(\tilde{\psi}_{k,m}\right)^{2}\left(\left(\tilde{\psi}_{k,m\prime}\right)^{*}\right)^{2}\rangle\right\} \tag{18}$$

It can be shown that

$$Re\left\{\langle\left(\tilde{\psi}_{k,m}\right)^{2}\left(\left(\tilde{\psi}_{k,m\prime}\right)^{*}\right)^{2}\rangle\right\} = 2Re\left\{\left(\langle v_{k,m,m\prime}\rangle\right)^{2}\right\} \tag{19}$$

The average value of the second term is:

$$\begin{aligned}\langle\left|\tilde{p}_{k,m}\right|^{2}\left|\tilde{p}_{k,m\prime}\right|^{2}\rangle \\ &= \langle\left|\tilde{\psi}_{k,m}\right|^{2}\left|\tilde{\psi}_{k,m\prime}\right|^{2}\rangle + \langle\left|\tilde{\psi}_{k,m}\right|^{2}\left|\tilde{p}_{z_{k,m\prime}}\right|^{2}\rangle \\ &+ \langle\left|\tilde{\psi}_{k,m\prime}\right|^{2}\left|\tilde{p}_{z_{k,m}}\right|^{2}\rangle + \langle\left|\tilde{p}_{z_{k,m}}\right|^{2}\left|\tilde{p}_{z_{k,m\prime}}\right|^{2}\rangle\end{aligned} \tag{20}$$

Define a Fourier transform beam power:

$$\tilde{P}_{k,m} = \sum_{q} \Delta\Omega_{\mathrm{pow}_{\mathrm{q}}} \frac{\langle kT_{sky}(\Omega_{\mathrm{q}})\rangle}{\lambda^{2}} \left|\tilde{\mathrm{a}}_{\mathrm{q,k,m}}\right|^{2} \tag{21}$$

and a Fourier transform noise power:

$$\tilde{P}_{z_m} = \sum_{n} \langle\left|p_{z_{n,m}}\right|^{2}\rangle = \sum_{n} \langle kT_{amp_{n,m}}\rangle \tag{22}$$

Then:

$$\langle \left|\tilde{p}_{k,m}\right|^2 \left|\tilde{p}_{k,m'}\right|^2 \rangle = 2\tilde{P}_{k,m}\tilde{P}_{k,m'} + \tilde{P}_{k,m}\tilde{P}_{z_{m'}} + \tilde{P}_{k,m'}\tilde{P}_{z_m} + \tilde{P}_{z_m}\tilde{P}_{z_{m'}} \tag{23}$$

The variation of the real part of the visibility is:

$$\begin{aligned}\langle \left(Re\left(v_{k,m,m'} - \langle v_{k,m,m'} \rangle\right)\right)^2 \rangle \\ = \tilde{P}_{k,m}\tilde{P}_{k,m'} + \frac{1}{2}\left(\tilde{P}_{k,m}\tilde{P}_{z_{m'}} + \tilde{P}_{k,m'}\tilde{P}_{z_m} + \tilde{P}_{z_m}\tilde{P}_{z_{m'}}\right) \\ + Re\left\{\left(\langle v_{k,m,m'} \rangle\right)^2\right\} - \left(Re\{\langle v_{k,m,m'} \rangle\}\right)^2\end{aligned} \tag{24}$$

In a similar fashion:

$$\begin{aligned}\langle \left(Im\left(v_{k,m,m'} - \langle v_{k,m,m'} \rangle\right)\right)^2 \rangle \\ = \tilde{P}_{k,m}\tilde{P}_{k,m'} + \frac{1}{2}\left(\tilde{P}_{k,m}\tilde{P}_{z_{m'}} + \tilde{P}_{k,m'}\tilde{P}_{z_m} + \tilde{P}_{z_m}\tilde{P}_{z_{m'}}\right) \\ - Re\left\{\left(\langle v_{k,m,m'} \rangle\right)^2\right\} - \left(Im\{\langle v_{k,m,m'} \rangle\}\right)^2\end{aligned} \tag{25}$$

Auto-Correlation of a Cylinder

The auto-correlation of a cylinder is:

$$\alpha_{k,m} = \left|\tilde{p}_{k,m}\right|^2 - \frac{1}{N}\sum_{k'=0}^{N-1}\left|\tilde{p}_{k',m}\right|^2 \tag{26}$$

Define:

$$\langle v_{k,m} \rangle = \sum_q \Delta\Omega_{\mathrm{pow}_\mathrm{q}} \frac{\langle kT_{sky}(\Omega_\mathrm{q}) \rangle}{\lambda^2} \left|\tilde{\mathrm{a}}_{\mathrm{q,k,m}}\right|^2 \tag{27}$$

Therefore:

$$\langle \alpha_{k,m} \rangle = \langle v_{k,m} \rangle - \frac{1}{N}\sum_{k'=0}^{N-1}\langle v_{k',m} \rangle \tag{28}$$

Following the same procedure as above

$$\langle \left(\alpha_{k,m} - \langle \alpha_{k,m} \rangle\right)^2 \rangle = \langle {\alpha_{k,m}}^2 \rangle - \langle \alpha_{k,m} \rangle^2 \tag{29}$$

After lots of algebra:

$$\begin{aligned}\langle \left(\alpha_{k,m} - \langle \alpha_{k,m} \rangle\right)^2 \rangle \\ = \left(\langle v_{k,m} \rangle - \frac{1}{N}\sum_{k'=0}^{N-1}\langle v_{k',m} \rangle\right)^2 + \left(\sum_n \langle kT_{amp_{n,m}} \rangle\right)^2 \\ + 2\langle v_{k,m} \rangle \sum_n \langle kT_{amp_{n,m}} \rangle \frac{N-1}{N}\end{aligned}$$