Signal to Noise for an FFT Antenna array

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Noise Distribution

In addition to the amplifier noise voltage, the sky signal also has characteristics of a thermal noise signal. In the frequency domain, the voltage can be described by a real and imaginary part from which the amplitude and phase can be calculated.

$$v_z = v_{z_R} + jv_{z_X} \tag{1}$$

Over many measurements, the mean value of the real and imaginary parts of a noise voltage is zero. However, the spread in the distribution is not zero and can be represented by a Gaussian distribution as shown in Figure 1 and Figure 2. Since the power is given by the magnitude squared of the real and imaginary parts of the voltage, the amplitude squared of the Gaussian distributions will produce an exponential distribution for the power as shown in Figure 3. The mean and the standard deviation of an exponential distribution are equal to each other.

$$\mu_P = \sigma_P = kT\Delta f = \sigma_R{}^2 + \sigma_X{}^2 \tag{2}$$

So that:

$$\sigma_R = \sigma_X = \sqrt{\frac{kT\Delta f}{2}} \tag{3}$$

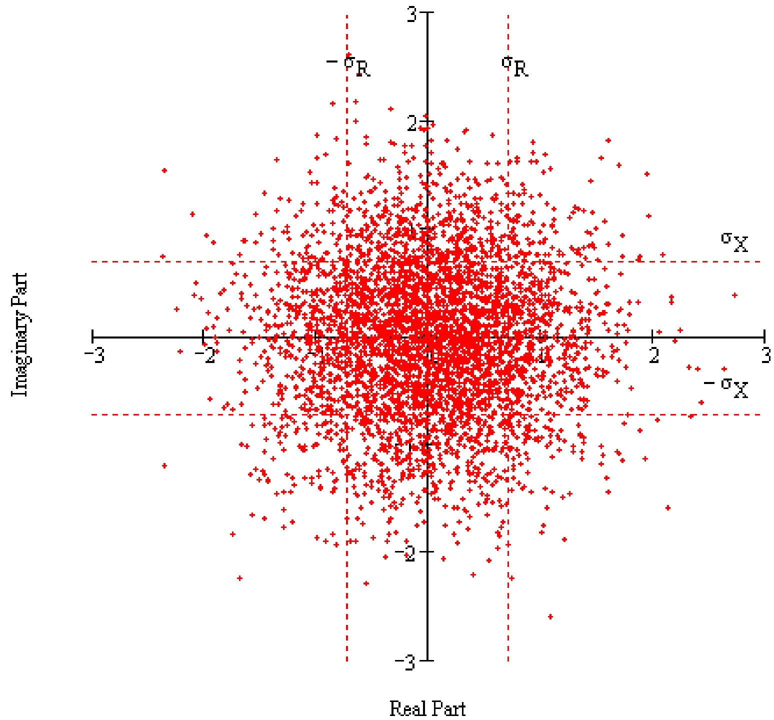

Figure 1. Example distribution many measurements of the real and imaginary parts of a noise voltage

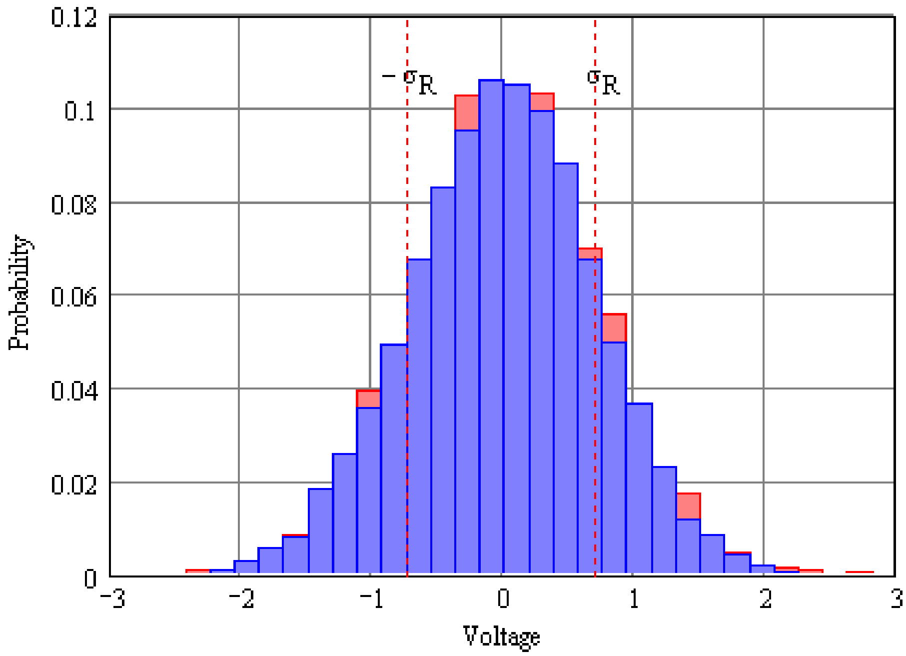

Figure 2. Histogram of the real(red) and imaginary(blue) parts of the distribution shown in Figure 1.

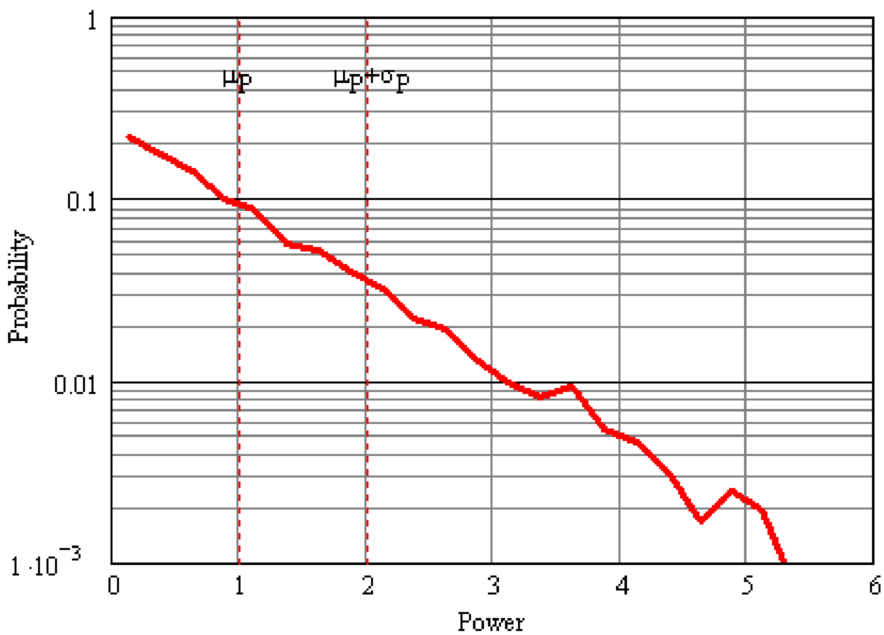

Figure 3. Histogram of power distribution calculated from the voltage distribution shown in Figure 1.

Array Voltage

The voltage from received by a single antenna feed in an FFT array is

$$v(x,\phi) = \sqrt{\frac{R_T}{\eta}}\vec{a}(x,\phi)\cdot\vec{E}(x,\phi) \tag{4}$$

where R_T is the termination resistance, η is the free space wave impedance, E is the incoming electric field, a is the antenna feed gain which has units of square root area, and

$$x = \sin(\theta) \tag{5}$$

where θ is the angle of incidence of the incoming wave onto the antenna array and ϕ is the azimuthal angle. The electric field can be rewritten as:

$$\vec{E}(x,\phi) = \sqrt{\eta}\vec{s}(x,\phi) \tag{6}$$

where **s** has units of square root power/area. The antenna feed voltage becomes:

$$\frac{v(x,\phi)}{\sqrt{R_T}} = \vec{a}(x,\phi)\cdot\vec{s}(x,\phi) \tag{7}$$

For a linear antenna array, the voltage at element n is

$$\frac{v_n(x,\phi)}{\sqrt{R_T}} = \vec{a}(x,\phi)\cdot\vec{s}(x,\phi)e^{-j2\pi n\frac{d}{\lambda}x} + \frac{v_{z_n}}{\sqrt{R_T}} \tag{8}$$

where d is the spacing between antenna feeds, λ is the wavelength, v_{zn} is the noise voltage from the amplifier located at the nth feed. By phase shifting the signal between antenna feeds a beam can be formed with its center at:

$$\sin(\theta_k) = x_k = \frac{k}{N}\frac{\lambda}{d} \tag{9}$$

where N is the total number of feeds. The voltage of this beam is given as:

$$\frac{V_k(x,\phi)}{\sqrt{R_T}} = \sum_{n=0}^{N-1}\left(\vec{a}(x,\phi)\cdot\vec{s}(x,\phi)e^{-j2\pi n\frac{d}{\lambda}x} + \frac{v_{z_n}}{\sqrt{R_T}}\right)e^{j2\pi\frac{n}{N}k} \tag{10}$$

To get the total voltage from all the sources in the sky, Equation 7 is integrated over the entire sky:

$$\frac{V_{k_T}}{\sqrt{R_T}} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\phi \int_{-1}^{1} \frac{V_k(x,\phi)}{\sqrt{R_T}}\,dx \tag{11}$$

For the purpose of this paper, assume a single point source:

$$\vec{s}(x,\phi) = \vec{s}_p\delta(x-x_p)\delta(\phi-\phi_p) \tag{12}$$

The total voltage becomes:

$$\frac{V_{k_T}}{\sqrt{R_T}} = \sum_{n=0}^{N-1}\left(\vec{a}_p\cdot\vec{s}_p e^{-j2\pi n\frac{d}{\lambda}x_p} + \frac{v_{z_n}}{\sqrt{R_T}}\right)e^{j2\pi\frac{n}{N}k} \tag{13}$$

Noise Distribution of Array Power

The power in this beam is:

$$P_k = \frac{\left|V_{k_T}\right|^2}{R_T} \tag{14}$$

The auto-correlation term of the above equation is:

$$P_A = \sum_{n=0}^{N-1} \left| \vec{a}_p \cdot \vec{s}_p e^{-j2\pi n \frac{d}{\lambda} x_p} + \frac{v_{z_n}}{\sqrt{R_T}} \right|^2 \tag{15}$$

To remove the effects of the amplifier noise, the auto-correlation term can be subtracted off:

$$\Delta P_k = \frac{\left|V_{k_T}\right|^2}{R_T} - P_A \tag{16}$$

The mean and the variation of the above equation can be determined by examining the number of independent pairs or baselines. The numbers of independent pairs of terms are:

$$N_p = \frac{1}{2} N(N-1) \tag{17}$$

Since the amplifier noise is uncorrelated with other amplifiers, the mean of Equation 16 is:

$$\mu_{\Delta k} = 2N_p \left| \vec{a}_p \cdot \vec{s}_p \right|^2 \tag{18}$$

The standard deviation does depend on the noise product of amplifiers

$$\sigma_{\Delta k} = 2N_p \left| \vec{a}_p \cdot \vec{s}_p \right|^2 + \sqrt{2N_p} kT_A + \sqrt{2N_p} \left| \vec{a}_p \cdot \vec{s}_p \right| \sqrt{kT_A} \tag{19}$$

Where the amplifier noise has been written as:

$$\frac{|v_z|^2}{R_T} = kT_A \tag{20}$$

The noise to signal becomes:

$$\frac{\sigma_{\Delta k}}{\mu_{\Delta k}} = 1 + \frac{1}{\sqrt{2N_p}} \frac{kT_A}{\left|\vec{a}_p \cdot \vec{s}_p\right|^2} + \frac{1}{\sqrt{2N_p}} \frac{\sqrt{kT_A}}{\left|\vec{a}_p \cdot \vec{s}_p\right|} \tag{21}$$

Two Dimensional Array

Assuming that the array is made of N_c cylinders in which each cylinder has a width W_c and a length L_c. In each cylinders there are N_f feeds spaced a distance d_f apart. The number of baselines along the length of the cylinder is:

$$N_{p_f} = \frac{1}{2} N_f (N_f - 1) \tag{22}$$

The number of baselines of cylinders is:

$$N_{p_c} = \frac{1}{2} N_c (N_c - 1) \tag{23}$$

The mean of the signal is:

$$\mu_{\Delta k} = 2N_{p_f} 2N_{p_c} \left| \vec{a}_p \cdot \vec{s}_p \right|^2 \tag{24}$$

The standard deviation is:

$$\sigma_{\Delta k} = 2N_p 2N_{p_c} \left| \vec{a}_p \cdot \vec{s}_p \right|^2 + \sqrt{2N_{p_f}} \sqrt{2N_{p_c}} \left(kT_A + \left| \vec{a}_p \cdot \vec{s}_p \right| \sqrt{kT_A} \right) \tag{25}$$

The noise to signal becomes:

$$\frac{\sigma_{\Delta k}}{\mu_{\Delta k}} = 1 + \frac{1}{2\sqrt{N_{p_f} N_{p_c}}} \left(\frac{kT_A}{\left|\vec{a}_p \cdot \vec{s}_p\right|^2} + \frac{\sqrt{kT_A}}{\left|\vec{a}_p \cdot \vec{s}_p\right|} \right) \tag{26}$$

Approximation of Pixel Signal Strength

The flux per polarization is given as:

$$\mathrm{I}(\sin(\theta), \phi) = \frac{\mathrm{kT_s}(\sin(\theta), \phi)}{\lambda^2} \tag{27}$$

The power density for a pixel is given as:

$$\mathrm{S_{pix}} = \int\limits_{\Delta\phi_{pix}} d\phi \int\limits_{\Delta\sin(\theta_{\mathrm{pix}})} \frac{\mathrm{kT_s}(\sin(\theta), \phi)}{\lambda^2} \mathrm{d}(\sin(\theta)) \tag{28}$$

The pixel size in declination is:

$$\Delta\sin(\theta_{\mathrm{pix}}) = \frac{\lambda}{L_{cyl}} = \frac{\lambda}{N_f d_f} \tag{29}$$

In case the cylinder packing is not uniform, there is a maximum cylinder location $\mathrm{N_L}$, so that the resolution in the azimuthal direction is:

$$\Delta\phi_{\mathrm{pix}} = \frac{\lambda}{N_L W_{cyl}} \tag{30}$$

Equation 33 becomes:

$$\mathrm{S_{pix}} = \left|\vec{\mathrm{s}}_{\mathrm{pix}}\right|^2 = \frac{\mathrm{kT_s}}{N_L N_f W_{cyl} d_f} \tag{31}$$

The effective area of a feed can be written as:

$$\mathrm{A_{f=}}|\vec{\mathrm{a}}_{\mathrm{f}}|^2 = \mathrm{h_f W_{cyl}} \tag{32}$$

For a feed composed of an infinitely short dipole:

$$\mathrm{h_f} \approx \lambda \tag{33}$$

So that the signal received by a single feed from a single pixel is:

$$\mathrm{S_{pix}A_{f=}}\left|\vec{\mathrm{a}}_{\mathrm{f}} \cdot \vec{\mathrm{s}}_{\mathrm{pix}}\right|^2 = \frac{1}{N_L N_f} \frac{\mathrm{h_f}}{d_f} \mathrm{kT_s} \tag{34}$$

Using Equation 31, the noise to signal becomes:

$$\frac{\sigma_{\Delta\mathrm{k}}}{\mu_{\Delta\mathrm{k}}} = 1 + \frac{\mathrm{h_f}}{d_f} \frac{\mathrm{T_A}}{\mathrm{T_s}} \frac{N_L}{N_C} \sqrt{\frac{N_C N_f}{(N_C - 1)(N_f - 1)}} + \sqrt{\frac{\mathrm{h_f}}{d_f} \frac{\mathrm{T_A}}{\mathrm{T_s}} \frac{N_L}{N_C}} \frac{1}{\sqrt{(N_C - 1)(N_f - 1)}} \tag{35}$$

In the limit of a large number of feeds,

$$\Delta\mathrm{T_s} = \mathrm{T_s} + \mathrm{T_A}\left[\frac{1}{p_f} \frac{\mathrm{h_f}}{d_f} \sqrt{\frac{N_C N_f}{(N_C - 1)(N_f - 1)}}\right] \tag{36}$$

If the signal is averaged over many measurements (M)

$$\Delta T_s = \frac{T_s + T_A\left[\frac{1}{p_f}\frac{h_f}{d_f}\sqrt{\frac{N_C N_f}{(N_C - 1)(N_f - 1)}}\right]}{\sqrt{M}} \tag{37}$$

where the number of measurements is given by:

$$M = \tau_{int}\Delta f \tag{38}$$

where τ_{int} is the integration time and Δf is the resolution bandwidth. The temperature resolution becomes: