Signal to Noise for an FFT Antenna array

Dave McGinnis
July 20, 2009

Noise Distribution

In addition to the amplifier noise voltage, the sky signal also has characteristics of a thermal noise signal. In the frequency domain, the voltage can be described by a real and imaginary part from which the amplitude and phase can be calculated.

$$v_z = v_{z_R} + jv_{z_X} \tag{1}$$

Over many measurements, the mean value of the real and imaginary parts of a noise voltage is zero. However, the spread in the distribution is not zero and can be represented by a Gaussian distribution as shown in Figure 1 and Figure 2. Since the power is given by the magnitude squared of the real and imaginary parts of the voltage, the amplitude squared of the Gaussian distributions will produce an exponential distribution for the power as shown in Figure 3. The mean and the standard deviation of an exponential distribution are equal to each other.

$$\mu_P = \sigma_P = kT\Delta f = {\sigma_R}^2 + {\sigma_X}^2 \tag{2}$$

So that:

$$\sigma_R = \sigma_X = \sqrt{\frac{kT\Delta f}{2}} \tag{3}$$

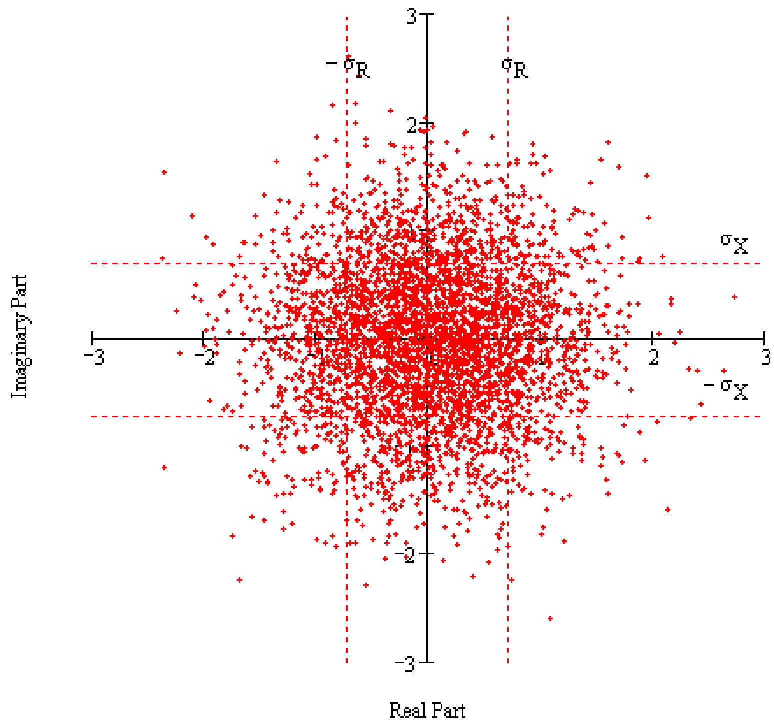

Figure 1. Example distribution many measurements of the real and imaginary parts of a noise voltage

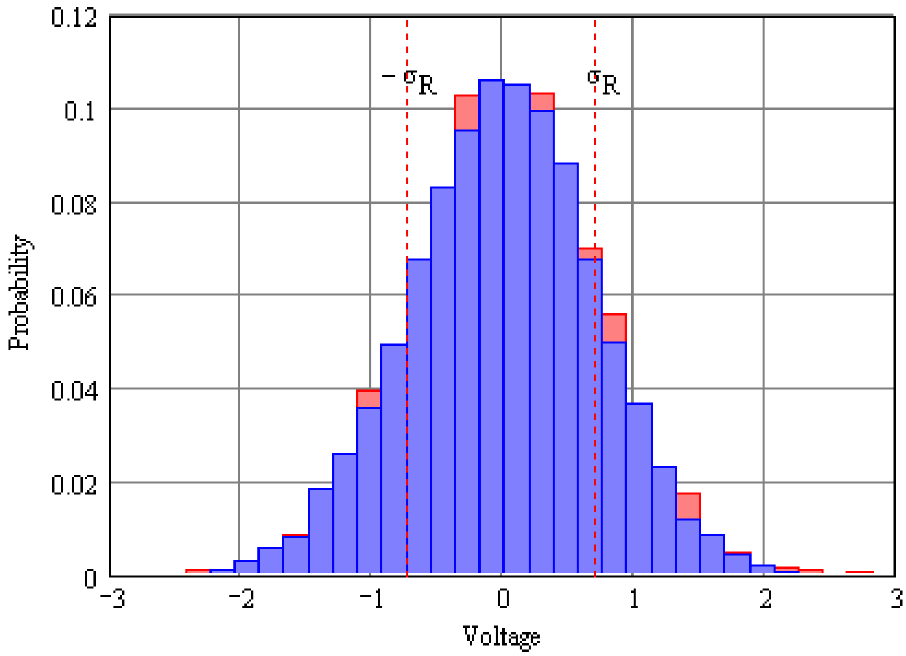

Figure 2. Histogram of the real(red) and imaginary(blue) parts of the distribution shown in Figure 1.

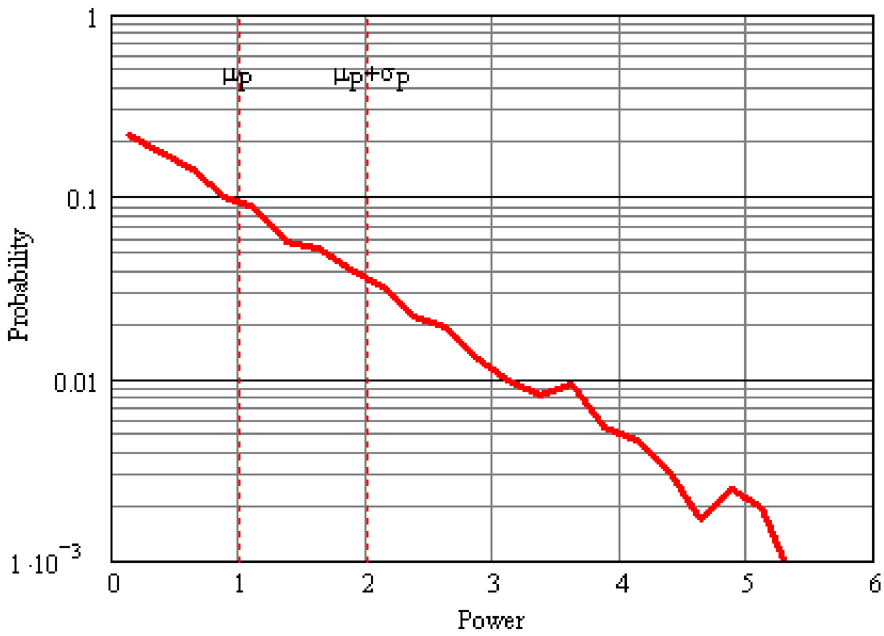

Figure 3. Histogram of power distribution calculated from the voltage distribution shown in Figure 1.

Array Voltage

The voltage from received by a single antenna feed in an FFT array is

$$v(x,\phi) = \sqrt{\frac{R_T}{\eta}}\vec{a}(x,\phi)\cdot\vec{E}(x,\phi) \tag{4}$$

where R_T is the termination resistance, η is the free space wave impedance, E is the incoming electric field, a is the antenna feed gain which has units of square root area, and

$$x = \sin(\theta) \tag{5}$$

where θ is the angle of incidence of the incoming wave onto the antenna array and ϕ is the azimuthal angle. The electric field can be rewritten as:

$$\vec{E}(x,\phi) = \sqrt{\eta}\vec{s}(x,\phi) \tag{6}$$

where **s** has units of square root power/area. The antenna feed voltage becomes:

$$\frac{v(x,\phi)}{\sqrt{R_T}} = \vec{a}(x,\phi)\cdot\vec{s}(x,\phi) \tag{7}$$

For a linear antenna array, the voltage at element n is

$$\frac{v_n(x,\phi)}{\sqrt{R_T}} = \vec{a}(x,\phi)\cdot\vec{s}(x,\phi)e^{-j2\pi n\frac{d}{\lambda}x} + \frac{v_{z_n}}{\sqrt{R_T}} \tag{8}$$

where d is the spacing between antenna feeds, λ is the wavelength, v_{zn} is the noise voltage from the amplifier located at the nth feed. By phase shifting the signal between antenna feeds a beam can be formed with its center at:

$$\sin(\theta_k) = x_k = \frac{k}{N}\frac{\lambda}{d} \tag{9}$$

where N is the total number of feeds. The voltage of this beam is given as:

$$\frac{V_k(x,\phi)}{\sqrt{R_T}} = \sum_{n=0}^{N-1}\left(\vec{a}(x,\phi)\cdot\vec{s}(x,\phi)e^{-j2\pi n\frac{d}{\lambda}x} + \frac{v_{z_n}}{\sqrt{R_T}}\right)e^{j2\pi\frac{n}{N}k} \tag{10}$$

To get the total voltage from all the sources in the sky, Equation 7 is integrated over the entire sky:

$$\frac{V_{k_T}}{\sqrt{R_T}} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\phi \int_{-1}^{1}\frac{V_k(x,\phi)}{\sqrt{R_T}}dx \tag{11}$$

For the purpose of this paper, assume a single point source:

$$\vec{s}(x,\phi) = \vec{s}_p\delta(x - x_p)\delta(\phi - \phi_p) \tag{12}$$

The total voltage becomes:

$$\frac{V_{k_T}}{\sqrt{R_T}} = \sum_{n=0}^{N-1}\left(\vec{a}_p\cdot\vec{s}_p e^{-j2\pi n\frac{d}{\lambda}x_p} + \frac{v_{z_n}}{\sqrt{R_T}}\right)e^{j2\pi\frac{n}{N}k} \tag{13}$$

The power in this beam is:

$$P_k = \frac{|V_{k_T}|^2}{R_T} \tag{14}$$

Equation 13 can be written as:

$$P_k = P_s + P_z + P_x \tag{15}$$

where the signal term P_s is

$$P_s = \sum_{n=0}^{N-1}\sum_{m=0}^{N-1}\left|\vec{a}_p \cdot \vec{s}_p\right|^2 e^{-j2\pi(n-m)\frac{d}{\lambda}x_p} e^{j2\pi\frac{n-m}{N}k} \quad (16)$$

The noise term P_z is:

$$P_z = \sum_{n=0}^{N-1}\sum_{m=0}^{N-1}\frac{v_{z_n} v_{z_m}^{*}}{R_T} e^{j2\pi\frac{n-m}{N}k} \quad (17)$$

The cross term P_x is:

$$P_x = \sum_{n=0}^{N-1}\sum_{m=0}^{N-1}\left(\left(\vec{a}_p \cdot \vec{s}_p\right) e^{-j2\pi n\frac{d}{\lambda}x_p}\frac{v_{z_m}^{*}}{\sqrt{R_T}} + \left(\vec{a}_p \cdot \vec{s}_p\right)^{*} e^{j2\pi m\frac{d}{\lambda}x_p}\frac{v_{z_n}}{\sqrt{R_T}}\right) e^{j2\pi\frac{n-m}{N}k} \quad (18)$$

Noise Distribution of Array Power

The mean of the noise distribution of the array can be determined by looking at the mean of the noise distributions of the individual power terms outlined in Equations 16-18. For the signal power term, P_s, the sky signal is completely correlated with all receivers so the mean will go as:

$$\mu_s = \left|\vec{a}_p \cdot \vec{s}_p\right|^2 N^2 \quad (19)$$

For the noise term, P_z, the amplifier noise between individual feeds is uncorrelated and cross products will average to zero.

$$\mu_z = \langle\frac{|v_z|^2}{R_T}\rangle N \quad (20)$$

For the cross term power, P_x, the sky signal and the amplifier noise is uncorrelated so the mean is zero.

$$\mu_x = 0 \quad (21)$$

The total mean becomes:

$$\mu_k = \mu_s + \mu_z + \mu_x \quad (22)$$

which reduces to:

$$\mu_k = \left|\vec{a}_p \cdot \vec{s}_p\right|^2 N^2 + \langle\frac{|v_z|^2}{R_T}\rangle N \quad (23)$$

Since the sky signal and the amplifier noise signal are both exponential distributions, the total noise distribution will also be exponential so that the standard deviation of the total distribution will be equal to the mean of the distribution.

$$\sigma_k = \mu_k \quad (24)$$

The average signal calculated in Equation 23 contains a large offset due to the amplifier noise. To remove the effects of the amplifier noise, the total array power can be subtracted off:

$$\Delta P_k = \frac{\left|V_{k_T}\right|^2}{R_T} - P_\Sigma \quad (25)$$

The power sum term, P_Σ is:

$$P_\Sigma = \sum_{n=0}^{N-1}\left|\vec{a}_p \cdot \vec{s}_p e^{-j2\pi n\frac{d}{\lambda}x_p} + \frac{v_{z_n}}{\sqrt{R_T}}\right|^2 \quad (26)$$

For the power sum term, P_Σ, there is no correlation between the sky and the amplifier so the mean is just the sum of both terms:

$$\mu_\Sigma = \left|\vec{a}_p \cdot \vec{s}_p\right|^2 N + \langle\frac{|v_z|^2}{R_T}\rangle N \tag{27}$$

The mean of the signal described by Equation 26 becomes:

$$\mu_{\Delta k} = \left|\vec{a}_p \cdot \vec{s}_p\right|^2 N(N-1) \tag{28}$$

The standard deviation of the subtracted signal becomes:

$$\sigma_{\Delta k} = \left|\vec{a}_p \cdot \vec{s}_p\right|^2 N(N-1) + \langle\frac{|v_z|^2}{R_T}\rangle (N-1) \tag{29}$$

The noise to signal for the array is:

$$\frac{\sigma_{\Delta k}}{\mu_{\Delta k}} = 1 + \frac{1}{N}\frac{\langle\frac{|v_z|^2}{R_T}\rangle}{\left|\vec{a}_p \cdot \vec{s}_p\right|^2} \tag{30}$$

For a two dimensional FFT array, the above analysis can be extended to:

$$\frac{\sigma_{\Delta k}}{\mu_{\Delta k}} = 1 + \frac{1}{N_\theta N_\phi}\frac{\langle\frac{|v_z|^2}{R_T}\rangle}{\left|\vec{a}_p \cdot \vec{s}_p\right|^2} \tag{31}$$

where N_θ is the number of rows of receivers aligned along the θ direction, and N_ϕ is the number of columns of receivers aligned along the ϕ direction.

Approximation of Pixel Signal Strength

The flux per polarization is given as:

$$I(\sin(\theta), \phi) = \frac{kT_s(\sin(\theta), \phi)}{\lambda^2} \tag{32}$$

The power density for a pixel is given as:

$$S_{pix} = \int_{\Delta\phi_{pix}} d\phi \int_{\Delta\sin(\theta_{pix})} \frac{kT_s(\sin(\theta), \phi)}{\lambda^2} d(\sin(\theta)) \tag{33}$$

Assuming that the array is made of N_{cyl} with width W_{cyl} with each cylinder is L_{cyl} long and that there are N_f feeds per cylinder spaced a distance d_f apart

$$\Delta\sin(\theta_{pix}) = \frac{\lambda}{L_{cyl}} = \frac{\lambda}{N_f d_f} \tag{34}$$

and:

$$\Delta\phi_{pix} = \frac{\lambda}{N_{cyl} W_{cyl}} \tag{35}$$

Equation 33 becomes:

$$S_{pix} = \left|\vec{s}_{pix}\right|^2 = \frac{kT_s(\sin(\theta_{pix}), \phi_{pix})}{N_{cyl} N_f W_{cyl} d_f} \tag{36}$$

The effective area of a feed composed of an infinitely short dipole placed at the focus of a cylinder is:

$$A_{f=}|\vec{a}_f|^2 = \lambda W_{cyl} \tag{37}$$

So that the signal received by a single feed from a single pixel is:

$$S_{pix}A_{f=}\left|\vec{a}_f \cdot \vec{s}_{pix}\right|^2 = \frac{1}{N_{cyl}N_f}\frac{\lambda}{d_f}\ kT_s\left(\sin\left(\theta_{pix}\right), \phi_{pix}\right) \tag{38}$$

Using Equation 31, the noise to signal becomes:

$$\frac{\sigma_{\Delta k}}{\mu_{\Delta k}} = 1 + \frac{d_f}{\lambda}\frac{kT_{amp}}{kT_s\left(\sin\left(\theta_{pix}\right), \phi_{pix}\right)} \tag{39}$$

where kT_{amp} is the amplifier noise power.

If the signal is averaged over many measurements (M)

$$\frac{\sigma_{\Delta k_M}}{\mu_{\Delta k}} = \frac{1}{\sqrt{M}}\left(1 + \frac{d_f}{\lambda}\frac{kT_{amp}}{kT_s\left(\sin\left(\theta_{pix}\right), \phi_{pix}\right)}\right) \tag{40}$$

where the number of measurements is given by:

$$M = \tau_{int}\Delta f \tag{41}$$

where τ_{int} is the integration time and Δf is the resolution bandwidth. The temperature resolution becomes:

$$\Delta T_s = \frac{T_s + \frac{d_f}{\lambda}T_{amp}}{\sqrt{\tau_{int}\Delta f}} \tag{42}$$