General Requirement Formulae for the 21cm Cylindrical Radio Telescope

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Introduction

This note will discuss a set of possible design parameters for the 21cm Cylindrical Radio telescope (CRT). The telescope concept consists of an array of cylindrical telescopes in which each cylinder is oriented along the north-south meridian.[1] Along the focus of each cylinder is a uniformly spaced array of antenna feeds. The antenna beams along the north-south meridian are formed by taking the spatial Fourier transform of the array of antenna feeds. The parameters that describe the (CRT) can be broken in to four categories:

1. Static Engineering Parameters (STE)
2. Dynamic Engineering Parameters (DYE)
3. Derived Engineering Parameters (DRE)
4. Scientific Parameters (SCI)

The static engineering parameters are independent parameters that are important in describing the telescope but are not easily changed for design optimization (such as the latitude of the telescope site, amplifier temperature, etc). Dynamic engineering parameters are independent parameters that can be easily varied during the design stage (such as feed spacing and the number of channels per cylinder.) Derived engineering parameters are design specific parameters such as cylinder length and width, but are derived from the static and dynamic engineering parameters. Scientific parameters are derived by the engineering parameters but the definition of the parameters themselves is not unique to any specific telescope design (such as angular resolution, or red-shift range).

Static Engineering Parameters

The desired frequency range of the CRT is in the neighborhood of 500-1000 MHz. The desired fraction bandwidth of 67% is too large to be handled by one frequency band. It will be assumed that the telescope consists of two separate frequency spans with a fractional bandwidth on the order of 33%. For simplicity, it will also be assumed that the frequency bands are adjacent to each other. In the future it might be desirable to have the frequency bands separated or overlapping. For cost saving purposes, it is envisioned that these frequency bands would not be measured simultaneously but sequentially. It is also assumed that the frequency bands would use the same reflectors and digital hardware. Table 1 shows the list of static engineering parameters. Most of the descriptions in the table are self-explanatory. The table includes a set of cost rates to estimate the cost of the telescope. It is not intended that these costs include everything that would arise in designing and building a large radio telescope, such as site preparation, non-recoverable engineering costs, overhead, contingency etc.,. These costs should only be used in trying to compare sets of design parameters.

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[1] "THE HUBBLE SPHERE HYDROGEN SURVEY, ", J Peterson, K. Bandura, U. Pen, arXiv:astro-ph/0606104 v1 6 Jun 2006

Number	Description	Symbol
STE.01	Survey Time	τ_s
STE.02	Observing Duty Factor	D_f
STE.03	Latitude of telescope site	α_L
STE.04	Average Sky Temperature	T_s
STE.05	Maximum Frequency Span per band	$\Delta F_{b_{max}}$
STE.06	Maximum Fractional Bandwidth per band	$\delta_{f\,b}$
STE.07	Number of Polarizations	N_p
STE.08	Antenna Feed Power Efficiency	g_a
STE.09	Cylinder Width / Cylinder Spacing	x_{cyl}
STE.10	Equivalent Amplifier Temperature	T_A
STE.11	Electronics Cost per Channel	R_e
STE.12	Feed Structure Cost per meter	R_f
STE.13	Reflector Cost per Cylinder volume	R_r

Table 1 Static engineering parameters

Dynamic Engineering Parameters

Number	Description	Symbol
DYE.01	Center Frequency of both bands combined	F_c
DYE.02	Average Feed Spacing	D_f
DYE.03	Number of digital channels per cylinder per polarization	N_f
DYE.04	Average Number of possible cylinder locations	N_L
DYE.05	Average Cylinder packing factor	p_f
DYE.06	Target Cost	C_T

Table 2 Dynamic Engineering Parameters

Derived Engineering Parameters

Number	Description	Symbol
DRE.01	Number of Cylinders	N_c
DRE.02	Cylinder Length	L_c
DRE.03	Cylinder Width	W_c
DRE.04	Cylinder Spacing	S_c
DRE.05	Declination Span	$\Delta\theta_d$
DRE.06	Feed Length	h_f
DRE.07	Feed Spacing	d_f
DRE.08	Band Center Frequency	$F_{c\,b}$
DRE.09	Wavelength	λ
DRE.10	Band Frequency Span	ΔF_{b}
DRE.11	Resolution Bandwidth	δf
DRE.12	Minimum Digital Memory	M_d
DRE.14	Integration Time per Pixel	τ_p
DRE.15	Number of Channels per polarization	N_{fT}
DRE.16	Electronics Cost	C_e
DRE.17	Feed Structure Cost	C_f
DRE.18	Reflector Cost	C_R
DRE.19	Total Cost	C_T

Table 3 Derived Engineering Parameters

Fractional Bandwidth

Because of limitations on digitizer bandwidth, it was assumed that there is a maximum frequency span that can be supported. Thus, because the frequency bands are adjacent, there is a limit to the fractional bandwidth so that the frequency span of the upper band does not exceed the maximum frequency span. The maximum fractional bandwidth that can be supported is

$$\delta_f < \frac{F_c}{\Delta F_{b_{max}}} \frac{2}{1 + \sqrt{\frac{4F_c - \Delta F_{b_{max}}}{2F_c}}} \tag{1}$$

The center frequency for the two bands are:

$$F_{c_\pm} = F_c \frac{4 \pm 2\delta_f}{4 + \delta_f{}^2} \tag{2}$$

The frequency span for the two bands are:

$$\Delta F_\pm = \delta_f F_{c_\pm} \tag{3}$$

Feed Spacing and the Number of Cylinder Locations

If the width and spacing of the cylinders are the same for both bands, and since the angular resolution of the telescope is inversely proportional to frequency, the number of cylinder locations should also be inversely proportional to frequency.

$$N_{L_\pm} = round\left(N_L \frac{F_c}{F_{c_\pm}}\right) \tag{4}$$

where the round function rounds to the nearest integer.

The angular resolution is given by the size of the telescope. If the angular resolution is constrained to be the same in both directions and the number of feeds per cylinder is the same for both bands, then the feed spacing should be proportional to the number of cylinder locations

$$d_{f_\pm} = D_f N_{L_\pm} \tag{5}$$

Packing Factor and Number of Cylinders

The number of cylinders is given by the packing factor. Because it makes sense to use the same number of feeds for both bands (if the digital electronics are to be re-used for each band) and the number of feeds per cylinder is the same, then the packing factor is given as:

$$p_{f_+} = p_{f_-} \frac{N_{L_-}}{N_{L_+}} \tag{6}$$

$$N_{f_+} = N_{f_-} = N_f \tag{7}$$

$$N_{C_+} = N_{C_+} = N_C = p_{f_-} N_{L_-} \tag{8}$$

However, we need to ensure that there enough cylinders to make p all the baselines. The Moffet redundancy factor for a linear array of cylinders is given as:[2]

$$R_\pm = \frac{1}{2} \frac{N_C(N_C - 1)}{N_{L_\pm} - 1} \tag{9}$$

[2] "Minimum-Redundancy Linear Array", A. Moffet, IEEE TRANSACTIONS ON ANTENNAS AND PROPAGATION, VOL. AP-16, NO. 2, MARCH 1968

For a minimally redundant array, R>1. Therefore:

$$N_C > \frac{1}{2}\left(1 + \sqrt{1 + 8(N_{L\_} - 1)}\right) \tag{10}$$

Or:

$$p_{f\_} > \frac{1}{2N_{L\_}}\left(1 + \sqrt{1 + 8(N_{L\_} - 1)}\right) \tag{11}$$

With these constraints, it is possible for p_{f+} to be greater than 1. A packing factor greater one means that the extra baselines can be used to increase the pixel sensitivity.

Cylinder Geometry

The cylinder length is given as:

$$L_{C_\pm} = N_f d_{f_\pm} \tag{12}$$

As was discussed earlier, the cylinder width and cylinder spacing should be constrained to be the same for both frequency bands.

$$W_C = x_{cyl} S_C = x_{cyl} \frac{N_f d_{f_\pm}}{N_{L_\pm}} \tag{13}$$

The declination span of the telescope is given as:

$$\sin\left(\frac{\Delta\theta_{d_\pm}}{2}\right) = \frac{\lambda}{2d_{f_\pm}} \tag{14}$$

The collecting area of each feed is given as:

$$\mathrm{A_f} = h_f W_C \tag{15}$$

Since the angular resolution of a cylinder in the width direction is λ/W_c , the declination span of an individual feed is:

$$\sin\left(\frac{\Delta\theta_f}{2}\right) = \frac{\lambda}{2h_f} \tag{16}$$

To make the declination span of the array match the declination span of an individual feed:

$$h_{f_\pm} = d_{f_\pm} \tag{17}$$

This is a specification on the feed design.

Telescope cost

As discussed earlier, it is not intended that these costs include everything that would arise in designing and building a large radio telescope, such as site preparation, non-recoverable engineering costs, overhead, contingency etc.,. These costs should only be used in trying to compare sets of design parameters. The cost of the digital electronics is assumed to scale only with the number of feeds:

$$C_e = N_f N_c N_p R_e \tag{18}$$

The cost of the telescope structure is broken into two parts. The feed line is the most complicated part of the reflector system and this cost will scale as the total length of the array.

$$C_f = L_c N_c R_f = N_f N_c d_f R_f \tag{19}$$

The cost of the main reflector surface will not only be proportional to area but height as well since tall structures will be more difficult to build. For a fixed f-ratio, the height will scale with cylinder width.

$$C_r = L_c N_c W_C^2 R_f = N_f N_c d_f W_C^2 R_f \tag{20}$$

The total cost is:

$$C_T = C_e + C_f + C_r \tag{21}$$

Scientific Parameters

Number	Description	Symbol
SCI.01	Maximum Red-shift	z_{max}
SCI.02	Minimum Red-shift	z_{min}
SCI.03	Angular Resolution	$\delta\psi$
SCI.04	Survey Area	A_s
SCI.05	Sensitivity per Pixel	δT_p
SCI.06	Figure of Merit with Plank Priors	FoM_p
SCI.07	Figure of Merit with Stage II Dark Energy Priors	FoM_{II}

Table 4

Red-shift Range

$$z_{min_\pm} = \frac{1.42GHz}{F_{c_\pm} - \frac{1}{2}\Delta F_\pm} - 1 \tag{22}$$

(23)

$$z_\pm = \frac{1.42GHz}{F_{c_\pm}} - 1 \tag{24}$$

$$z_{max_\pm} = \frac{1.42GHz}{F_{c_\pm} + \frac{1}{2}\Delta F_\pm} - 1 \tag{25}$$

Angular Resolution

$$sin(\delta\psi_\pm) = \frac{\lambda}{N_f d_{f_\pm}} \tag{26}$$

Resolution Bandwidth

For physics reasons, it was decided to set the resolution bandwidth to match the angular resolution pixel size. The resolution in red-shift determines the depth of the 3-D pixel. An empirical formula for a uniform 3-D pixel is:

$$\delta z_\pm \approx 0.436 \times \delta\psi_\pm(\text{radians}) \times z_\pm(z_\pm + 2) \tag{27}$$

The frequency resolution is as a function of red-shift resolution is:

$$\delta f_\pm = \frac{1.4GHz}{(1 + z_\pm)^2}\delta z_\pm \tag{28}$$

which becomes

$$\delta f_{\pm} \approx 610 MHz \times \delta\theta(\text{radians}) \times \frac{z_{\pm}(z_{\pm}+2)}{(1+z_{\pm})^2} \tag{29}$$

The digital memory required is:

$$M_{d_{\pm}} = \frac{2\Delta F_{\pm}}{\delta f_{\pm}} \tag{30}$$

Survey Area

The survey area for a drift telescope:

$$A = \int_0^{2\pi} d\phi \int_{\theta_{d_{min}}}^{\theta_{d_{max}}} \cos(\theta)\, d\theta = 2\pi\left[sin\left(\theta_{d_{max}}\right) - sin\left(\theta_{d_{min}}\right)\right] \tag{31}$$

where θ_d is the declination reach of the telescope. The declination reach of the telescope is given by:

$$\begin{aligned} \theta_{d_{max}} &= \alpha_L + \frac{\Delta\theta_d}{2} \quad if \quad \alpha_L + \frac{\Delta\theta_d}{2} < \frac{\pi}{2} \\ \theta_{d_{max}} &= \frac{\pi}{2} \quad if \quad \alpha_L + \frac{\Delta\theta_d}{2} > \frac{\pi}{2} \end{aligned} \tag{32}$$

And:

$$\begin{aligned} \theta_{d_{min}} &= \alpha_L - \frac{\Delta\theta_d}{2} \quad if \quad \alpha_L - \frac{\Delta\theta_d}{2} < -\frac{\pi}{2} \\ \theta_{d_{min}} &= -\frac{\pi}{2} \quad if \quad \alpha_L - \frac{\Delta\theta_d}{2} < -\frac{\pi}{2} \end{aligned} \tag{33}$$

Integration Time and Pixel Sensitivity

To calculate the integration time, it is necessary to calculate the amount of time a point object spends in the beam of a single cylinder. This time will be a function of the declination coordinate of the object. For example an object at the north celestial pole never leaves the cylinder beam. The array of N_f feeds along the length of a cylinder forms N_f beams. The angle of the beam **n** from vertical along the meridian direction is given as:

$$\sin(\psi_n) = \left(-\frac{1}{2} + \frac{n}{N_f}\right)\frac{\lambda}{d_f} \tag{34}$$

The declination of beam n is:

$$\theta_n = \psi_n + \alpha_L \tag{35}$$

The beam width of a single cylinder is

$$\Delta\phi_C = \frac{\lambda}{W_c} \tag{36}$$

The span in hour angle is:

$$\sin\left(\frac{\Delta\varphi_{RA_n}}{2}\right) = \frac{1}{cos(\theta_n)}\sin\left(\frac{\Delta\phi_C}{2}\right) \tag{37}$$

If the magnitude of the right hand side of the above equation is greater than one, then the object never leaves the cylinder beam and $\Delta\phi_{RAn}$ is limited to +/- π. The average integration time is:

$$\tau_p = \frac{\tau_s D_f}{N_f + 1} \sum_{n=0}^{N_f} \frac{\Delta\varphi_{RA_n}}{2\pi} \tag{38}$$

The pixel sensitivity for a given band is given as:

$$\delta \mathrm{T}_{\mathrm{p}_{\pm}} = \frac{1}{\sqrt{\tau_{p_{\pm}} \delta f_{\pm}}} \left(\mathrm{T_s} + \frac{1}{\mathrm{g_a}} \frac{1}{p_{f_{\pm}}} \frac{d_{f_{\pm}}}{\mathrm{h}_{\mathrm{f}_{\pm}}} \sqrt{\frac{N_f}{(N_f - 1)}} \sqrt{\frac{N_c}{(N_c - 1)}} \mathrm{T_A} \right) \tag{39}$$