Antenna Factor for the 21 cm Simulation

Dave McGinnis
November 29, 2008

Antenna Response

To simulate the response of an antenna, an incoming plane wave of the form:

$$\vec{E}_{inc}(\vec{k}) = \left(\vec{E}_{ra}(\vec{k}) + \vec{E}_{dec}(\vec{k})\right) e^{j\vec{k}\cdot\vec{r}} e^{j\omega t} \quad (1)$$

impinges on the antenna where the subscript "**ra**" denotes polarization along right ascension axis and the subscript "**dec**" notes polarization along the declination axis. The direction of the incoming wave is denoted by the vector **k** because a time dependence of +jωt is used. If the antenna is located at **r**=0, the voltage at the output terminals of the antenna is given as:

$$v_o(\vec{k}) = g_{ra}(\vec{k})\left(\vec{E}_{ra}(\vec{k}) \cdot \hat{\theta}_{ra}\right) + g_{dec}(\vec{k})\left(\vec{E}_{dec}(\vec{k}) \cdot \hat{\phi}_{dec}\right) \quad (2)$$

For an antenna located at **r**

$$v(r, \vec{k}) = v_o(\vec{k}) e^{j\vec{k}\cdot\vec{r}} \quad (3)$$

Assume that the output terminal of antenna is terminated with a resistance R_T. Define the following terms:

$$a_{ra}(\vec{k}) = \sqrt{\frac{\eta}{R_T}}\, g_{ra}(\vec{k}) \quad (4)$$

and

$$a_{dec}(\vec{k}) = \sqrt{\frac{\eta}{R_T}}\, g_{dec}(\vec{k}) \quad (5)$$

where $\mathbf{a_{ra}}$ and $\mathbf{a_{dec}}$ have units of m. Also:

$$s_{ra}(\vec{k}) = \frac{1}{\sqrt{2\eta}} \vec{E}_{ra}(\vec{k}) \cdot \hat{\theta}_{ra} \quad (6)$$

and:

$$s_{dec}(\vec{k}) = \frac{1}{\sqrt{2\eta}} \vec{E}_{dec}(\vec{k}) \cdot \hat{\phi}_{dec} \quad (7)$$

where $\mathbf{s_{ra}}$ and $\mathbf{s_{dec}}$ have units of Watts$^{1/2}$/m. Equation 2 becomes:

$$\frac{v_o(\vec{k})}{\sqrt{2R_T}} = a_{ra}(\vec{k}) s_{ra}(\vec{k}) + a_{dec}(\vec{k}) s_{dec}(\vec{k}) \quad (8)$$

Radiation Flux

The radiation from the sky can be characterized in terms of intensity per steradian:

$$I = 2\frac{k_B T_s}{\lambda^2} \quad (9)$$

The intensity I is measured in Watts/m^2/Hz/steradian. This can be taken as the definition of the temperature, but a blackbody radiator at temperature T_s would generate this flux per steradian per meter squared as seen from the receiving antenna. The intensity described in Equation 9 is for the sum of both modes of polarization. For a single mode of polarization, we can define:

$$s_{ra}(\vec{k}) = \frac{\sqrt{k_B \Delta\upsilon}}{\lambda} \tau_{ra}(\vec{k}) \tag{10}$$

and

$$s_{dec}(\vec{k}) = \frac{\sqrt{k_B \Delta\upsilon}}{\lambda} \tau_{dec}(\vec{k}) \tag{11}$$

where $\Delta\nu$ is a bin of frequency bandwidth and $\boldsymbol{\tau_{ra}}$ and $\boldsymbol{\tau_{dec}}$ have units of Kelvin$^{1/2}$. Equation 8 becomes:

$$\frac{v_o(\vec{k})}{\sqrt{2R_T}} = \frac{\sqrt{k_B \Delta\upsilon}}{\lambda} \left(a_{ra}(\vec{k})\tau_{ra}(\vec{k}) + a_{dec}(\vec{k})\tau_{dec}(\vec{k}) \right) \tag{12}$$

The total voltage at the output of the antenna is the integral overall all possible angles:

$$v_{o_T} = \iint_{\Omega_k} v_o(\vec{k}) d\Omega_k \tag{13}$$

However, the power received in a bandwidth $\Delta\nu$ by the antenna is given as:

$$P = \frac{|v_{o_T}|^2}{2R_T} = \frac{1}{2R_T} \iint_{\Omega_{k_1}} \iint_{\Omega_{k_2}} v_o(\vec{k}_1) v_o^*(\vec{k}_2) d\Omega_{k_1} d\Omega_{k_2} \tag{14}$$

Since the sky signal is noise, the sky signal at different angles and different polarizations is uncorrelated. The power in sky bin $d\Omega$ is given as:

$$dP_{ra} = \frac{k_B \Delta\upsilon}{\lambda^2} |a_{ra}(\vec{k})|^2 |\tau_{ra}(\vec{k})|^2 d\Omega_k \tag{15}$$

and:

$$dP_{dec} = \frac{k_B \Delta\upsilon}{\lambda^2} |a_{dec}(\vec{k})|^2 |\tau_{dec}(\vec{k})|^2 d\Omega_k \tag{16}$$

where $\mathbf{a_{ra}}$ and $\mathbf{a_{dec}}$ are given by Equations 4-5. The total power is:

$$P = \iint_{\Omega_k} dP_{ra} + \iint_{\Omega_k} dP_{dec} \tag{17}$$

Simple example

Consider a parabolic antenna with its long axis aligned to the x direction. The wave vector is given as:

$$\vec{k} = k_x \hat{x} + k_y \hat{y} + k_z \hat{z} \tag{18}$$

The spherical coordinate system is defined as:

$$\begin{aligned} k_x &= |\vec{k}| \sin(\theta) \\ k_y &= |\vec{k}| \cos(\theta) \sin(\phi) \\ k_z &= |\vec{k}| \cos(\theta) \cos(\phi) \end{aligned} \tag{19}$$

For a short dipole oriented along the x direction which is mostly sensitive to the declination polarization, the normalized radiation pattern in the θ direction is:

$$R_{\theta_{dec}}(\theta) = \cos(\theta) \tag{20}$$

The parabolic cylinder focuses the field with a pattern given approximately by:

$$R_{\phi_{dec}}(\phi) = \frac{\sin\left(\pi \frac{W}{\lambda} \sin(\phi)\right)}{\pi \frac{W}{\lambda} \sin(\phi)} \tag{21}$$

Where W is the width of the cylinder in the **y** direction. The beam area of the main lobe is given as:

$$\Omega_{A_{dec}} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\sin^{-1}\left(\frac{\lambda}{W}\right)}^{\sin^{-1}\left(\frac{\lambda}{W}\right)} \left(R_{\theta_{dec}}(\theta) R_{\phi_{dec}}\right)^2 \cos(\theta) d\theta d\phi \tag{22}$$

which is approximately equal to:

$$\Omega_{A_{dec}} \approx 1.2 \frac{\lambda}{W} \tag{23}$$

Since the total effective area is:

$$A_{dec} = \frac{\lambda^2}{\Omega_{A_{dec}}} \tag{24}$$

then

$$A_{dec} = \frac{\lambda W}{1.2} \tag{25}$$

giving:

$$a_{dec}(\theta, \phi) = \sqrt{\frac{\lambda W}{1.2}} \frac{\sin\left(\pi \frac{W}{\lambda} \sin(\phi)\right)}{\pi \frac{W}{\lambda} \sin(\phi)} \cos(\theta) \tag{26}$$

For the right ascension polarization, the dipole feed would be pointed along the y direction so the normalized radiation pattern becomes:

$$R_{\theta_{ra}} = 1 \tag{27}$$

In the ϕ direction:

$$R_{\phi_{ra}} = \frac{\sin\left(\pi \frac{W}{\lambda} \sin(\phi)\right)}{\pi \frac{W}{\lambda} \sin(\phi)} \cos(\phi) \tag{28}$$

The beam area for the RA polarization:

$$\Omega_{A_{ra}} \approx 2.835 \frac{\lambda}{W} \tag{29}$$

then

$$A_{dec} = \frac{\lambda W}{2.835} \tag{30}$$

giving:

$$a_{ra}(\theta,\phi) = \sqrt{\frac{\lambda W}{2.835}} \frac{\sin\left(\pi \frac{W}{\lambda} \sin(\phi)\right)}{\pi \frac{W}{\lambda} \sin(\phi)} \cos(\phi) \tag{31}$$

Summary

The power in sky bin $d\Omega$ is given as:

$$dP_{ra} = \frac{k_B \Delta\upsilon}{\lambda^2} \left|a_{ra}(\vec{k})\right|^2 \left|\tau_{ra}(\vec{k})\right|^2 d\Omega_k \tag{32}$$

and:

$$dP_{dec} = \frac{k_B \Delta\upsilon}{\lambda^2} \left|a_{dec}(\vec{k})\right|^2 \left|\tau_{dec}(\vec{k})\right|^2 d\Omega_k \tag{33}$$

Where $\mathbf{a_{ra}}$ and $\mathbf{a_{dec}}$ have units of meters and $\boldsymbol{\tau_{ra}}$ and $\boldsymbol{\tau_{dec}}$ have units of $K^{1/2}$. For a parabolic cylinder antenna with an infinitely short dipole feed:

$$a_{ra}(\theta,\phi) = \sqrt{\frac{\lambda W}{2.835}} \frac{\sin\left(\pi \frac{W}{\lambda} \sin(\phi)\right)}{\pi \frac{W}{\lambda} \sin(\phi)} \cos(\phi) \tag{34}$$

and:

$$a_{dec}(\theta,\phi) = \sqrt{\frac{\lambda W}{1.2}} \frac{\sin\left(\pi \frac{W}{\lambda} \sin(\phi)\right)}{\pi \frac{W}{\lambda} \sin(\phi)} \cos(\theta) \tag{35}$$

where the incoming wave has a wave vector:

$$\vec{k} = k_x\hat{x} + k_y\hat{y} + k_z\hat{z} \tag{36}$$

and:

$$\begin{aligned} k_x &= \left|\vec{k}\right|\sin(\theta) \\ k_y &= \left|\vec{k}\right|\cos(\theta)\sin(\phi) \\ k_z &= \left|\vec{k}\right|\cos(\theta)\cos(\phi) \end{aligned} \tag{37}$$

If the antenna is not located at the origin but at position **r**:

$$a\left(r, \vec{k}(\theta,\phi)\right) = a\left(\vec{k}(\theta,\phi)\right) e^{j\vec{k}(\theta,\phi)\cdot\vec{r}} \tag{38}$$