

# Spherical Coordinates for a Parabolic Cylinder Antenna

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## Introduction

The 21 cm project is based on a parabolic cylinder antenna with receivers placed periodically along the focus line of the antenna. This note proposes a set of coordinate systems to use when calculating the polarization pattern of the antenna. The standard XYZ coordinate system that is presently in use for describing the antenna is shown in Figure 1. The telescope is aligned along the x-axis with the focus line of the antenna coinciding with the x axis. The parabola points up along the Z axis. For describing the antenna pattern, spherical coordinates are used. Typical spherical coordinate system have a polar angle referenced to the Z axis and an azimuthal angle for describing projections onto the XY plane as shown in *Figure 2*.

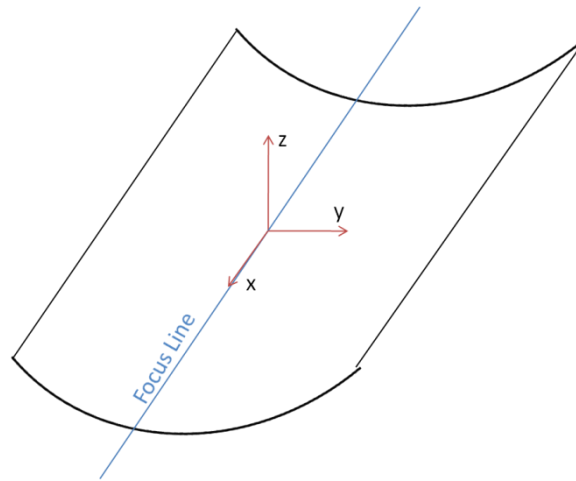


Figure 1. Coordinate system of the antenna cylinder

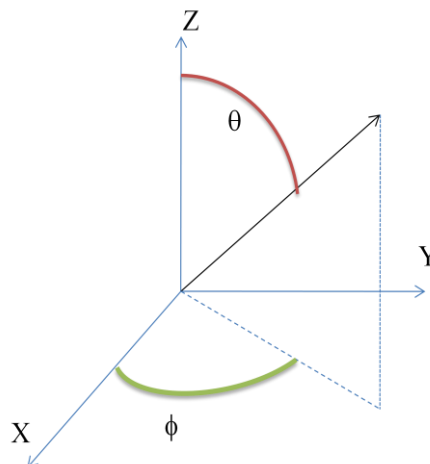


Figure 2. Spherical coordinate system referenced to the Z axis.

## Directivity

The directivity of an antenna is defined as the ratio of the radiation intensity to average radiation intensity:

$$D(\phi, \theta) = \frac{U(\phi, \theta)}{\frac{1}{4\pi} \iint U(\phi, \theta) \sin(\theta) d\theta d\phi} \quad (1)$$

where the radiation intensity is given as::

$$U(\phi, \theta) = \frac{1}{2} \text{Re}\{\vec{E} \times \vec{H}^*\} \cdot \hat{r} R^2 \quad (2)$$

For a parabolic cylinder antenna, the radiation pattern varies rapidly through the X-Z plane. The element angular surface area,  $\sin(\theta)d\theta d\phi$ , also varies rapidly though the X-Z plane because of the  $\sin(\theta)$  dependence. This makes defining a mesh for the numerical evaluation of the integral in Equation (1) difficult. The mesh is easier to define if the polar angle,  $\theta$ , is referenced to the Y axis, and the azimuthal angle,  $\phi$ , is referenced to projections in the XZ plane as shown in Figure 3.

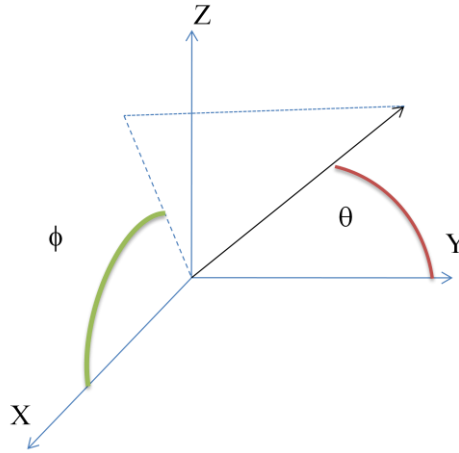


Figure 3. Spherical coordinate system referenced to the Y axis

The unit vectors for the spherical coordinate system referenced to the Y axis as plane as shown in Figure 3 are given as:

$$\begin{bmatrix} \hat{r} \\ \hat{\theta} \\ \hat{\phi} \end{bmatrix} = \begin{bmatrix} \sin(\theta)\sin(\phi) & \cos(\theta) & \sin(\theta)\cos(\phi) \\ \cos(\theta)\sin(\phi) & -\sin(\theta) & \cos(\theta)\cos(\phi) \\ \cos(\phi) & 0 & -\sin(\phi) \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} \quad (3)$$

## Polarization

Because the 21 cm telescope will be used in the drift scan mode, it makes more sense to define the polarization with respect to the celestial coordinate system. For the drift scan mode, the telescope will be oriented along a north-south line. The altitude of the celestial pole ( $Z_p$  axis) will be equal to the latitude of the telescope location as shown in Figure 4. The transformation between the telescope coordinate system and the celestial coordinate system is given as:

$$\begin{bmatrix} X_p \\ Y_p \\ Z_p \end{bmatrix} = \begin{bmatrix} \sin(\alpha) & 0 & \cos(\alpha) \\ 0 & 1 & 0 \\ -\cos(\alpha) & 0 & \sin(\alpha) \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \quad (4)$$

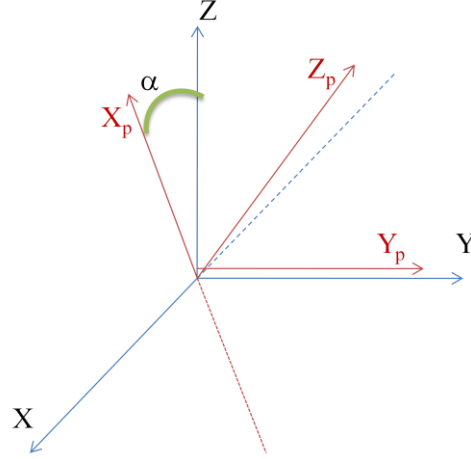


Figure 4. Celestial coordinate system (in red) with respect to telescope coordinate system. The angle  $\alpha$  equal to the latitude of the telescope location

The celestial coordinate system is defined as by a declination angle,  $\theta_d$ , and a right ascension angle,  $\phi_r$ , as shown in Figure 5.

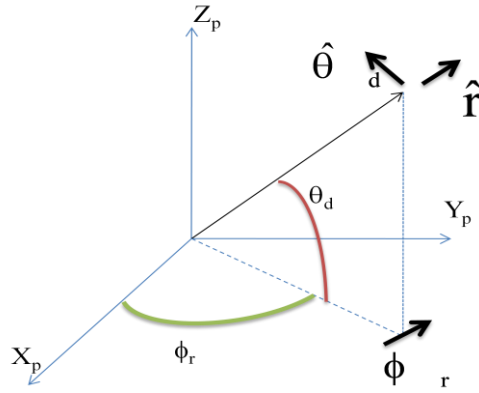


Figure 5. Celestial Coordinate system defined by declination angle,  $\theta_d$ , and right ascension angle,  $\phi_r$ .

The declination and right ascension unit vectors are given as:

$$\begin{bmatrix} \hat{r}_p \\ \hat{\theta}_d \\ \hat{\phi}_r \end{bmatrix} = \begin{bmatrix} \cos(\theta_d)\cos(\phi_r) & \cos(\theta_d)\sin(\phi_r) & \sin(\theta_d) \\ -\sin(\theta_d)\cos(\phi_r) & -\sin(\theta_d)\sin(\phi_r) & \cos(\theta_d) \\ -\sin(\phi_r) & \cos(\phi_r) & 0 \end{bmatrix} \begin{bmatrix} \hat{x}_p \\ \hat{y}_p \\ \hat{z}_p \end{bmatrix} \quad (5)$$

The inverse of Equation 5 is:

$$\begin{bmatrix} \widehat{x}_p \\ \widehat{y}_p \\ \widehat{z}_p \end{bmatrix} = \begin{bmatrix} \cos(\theta_d)\cos(\phi_r) & -\sin(\theta_d)\cos(\phi_r) & -\sin(\phi_r) \\ \cos(\theta_d)\sin(\phi_r) & -\sin(\theta_d)\sin(\phi_r) & \cos(\phi_r) \\ \sin(\theta_d) & \cos(\theta_d) & 0 \end{bmatrix} \begin{bmatrix} \widehat{r}_p \\ \widehat{\theta}_d \\ \widehat{\phi}_r \end{bmatrix} \quad (6)$$

In turn, the relationship between the Cartesian unit vectors in the celestial and telescope coordinate systems is given by the inverse of Equation 4:

$$\begin{bmatrix} \widehat{x} \\ \widehat{y} \\ \widehat{z} \end{bmatrix} = \begin{bmatrix} \sin(\alpha) & 0 & -\cos(\alpha) \\ 0 & 1 & 0 \\ \cos(\alpha) & 0 & \sin(\alpha) \end{bmatrix} \begin{bmatrix} \widehat{x}_p \\ \widehat{y}_p \\ \widehat{z}_p \end{bmatrix} \quad (7)$$

Combining Equations 6 and 7:

$$\begin{bmatrix} \widehat{x} \\ \widehat{y} \\ \widehat{z} \end{bmatrix} = \begin{bmatrix} \sin(\alpha) & 0 & -\cos(\alpha) \\ 0 & 1 & 0 \\ \cos(\alpha) & 0 & \sin(\alpha) \end{bmatrix} \begin{bmatrix} \cos(\theta_d)\cos(\phi_r) & -\sin(\theta_d)\cos(\phi_r) & -\sin(\phi_r) \\ \cos(\theta_d)\sin(\phi_r) & -\sin(\theta_d)\sin(\phi_r) & \cos(\phi_r) \\ \sin(\theta_d) & \cos(\theta_d) & 0 \end{bmatrix} \begin{bmatrix} \widehat{r}_p \\ \widehat{\theta}_d \\ \widehat{\phi}_r \end{bmatrix} \quad (8)$$

The polarization unit vector for the declination angle is:

$$\begin{aligned} \widehat{\theta}_d = & -(\sin(\alpha)\sin(\theta_d)\cos(\phi_r) + \cos(\alpha)\cos(\theta_d))\widehat{x} \\ & +(\sin(\theta_d)\sin(\phi_r))\widehat{y} \\ & +(-\cos(\alpha)\sin(\theta_d)\cos(\phi_r) + \sin(\alpha)\cos(\theta_d))\widehat{z} \end{aligned} \quad (9)$$

The polarization unit vector for the right ascension angle is:

$$\begin{aligned} \widehat{\phi}_r = & -(\sin(\alpha)\sin(\phi_r))\widehat{x} \\ & +(\cos(\phi_r))\widehat{y} \\ & -(\cos(\alpha)\sin(\phi_r))\widehat{z} \end{aligned} \quad (10)$$

The cosine and sine of the declination and right ascension angles are related to the telescope based angles by:

$$\cos(\theta_d) = \frac{\sqrt{x_p^2 + y_p^2}}{\sqrt{x_p^2 + y_p^2 + z_p^2}} \quad (11)$$

$$\sin(\theta_d) = \frac{z_p}{\sqrt{x_p^2 + y_p^2 + z_p^2}} \quad (12)$$

$$\cos(\phi_r) = \frac{x_p}{\sqrt{x_p^2 + y_p^2}} \quad (13)$$

$$\sin(\phi_r) = \frac{y_p}{\sqrt{x_p^2 + y_p^2}} \quad (14)$$

Using Equations 3 and 4:

$$\begin{bmatrix} x_p \\ y_p \\ z_p \end{bmatrix} = \begin{bmatrix} \sin(\theta)\cos(\alpha - \phi) \\ \cos(\theta) \\ \sin(\theta)\sin(\alpha - \phi) \end{bmatrix} \quad (15)$$

### **Example**

Equations 9-15 show that the declination and right ascension polarization will be a function of the telescope's latitude. Figures 6-9 show simulations for a 10 meter wide x 30 meter long cylinder with a focus of 2.5 meters at 1 GHz. The feed is a single dipole that is  $\lambda/2$  long. The simulations show the polarization magnitudes for various telescope latitudes and feed orientation.

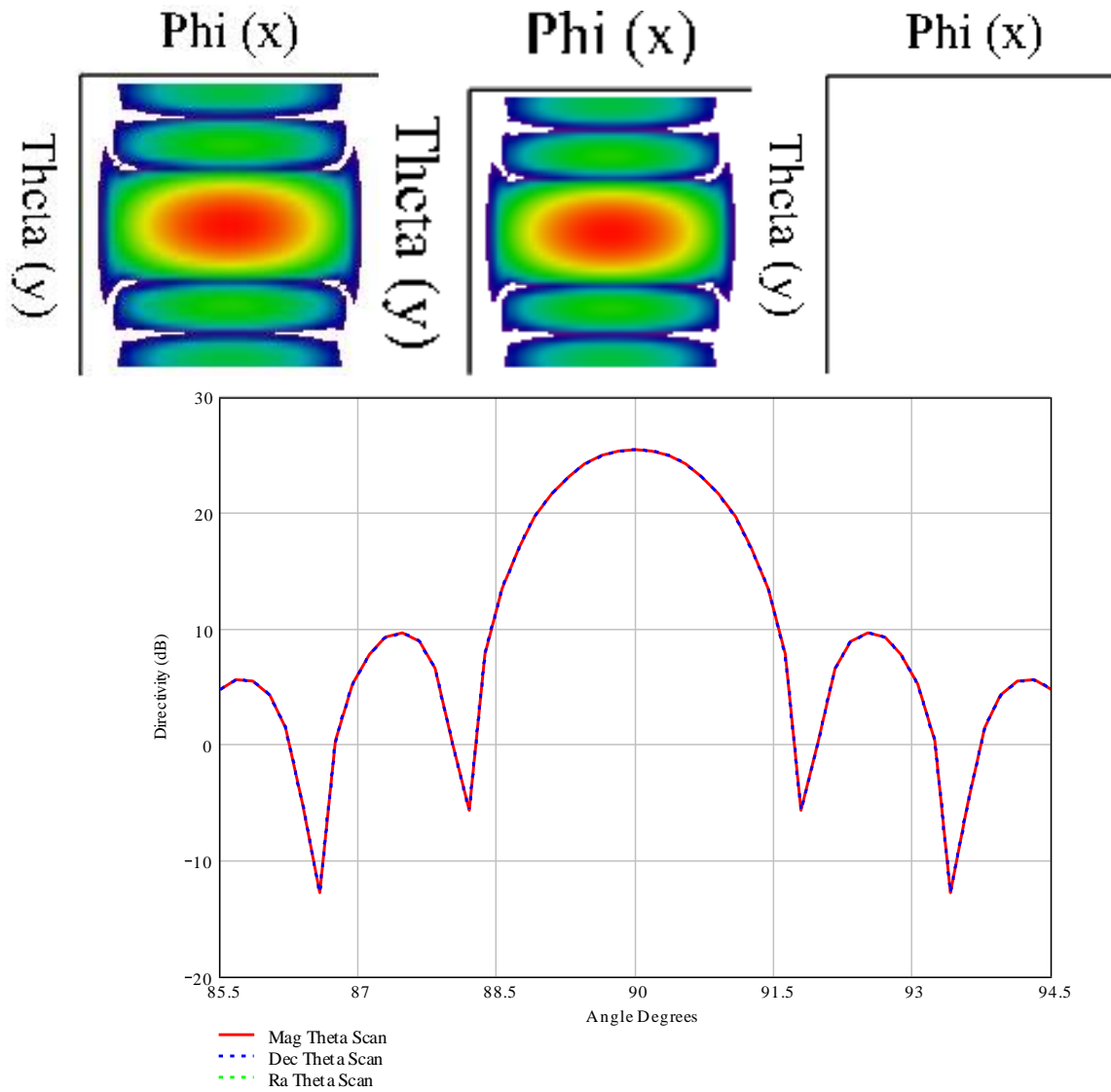


Figure 6. Simulation of a 10 meter wide x 30 meter long cylinder with a focus of 2.5 meters at 1 GHz. The feed is a single dipole oriented in the x direction that is  $\lambda/2$  long. The latitude of the telescope is 0 degrees. The color contour plots have 40 dB of range. The left plot is the magnitude, the middle plot is the declination polarization and the left plot is the right ascension polarization. The graph is a  $\theta$  scan for  $\phi=0$  degrees.

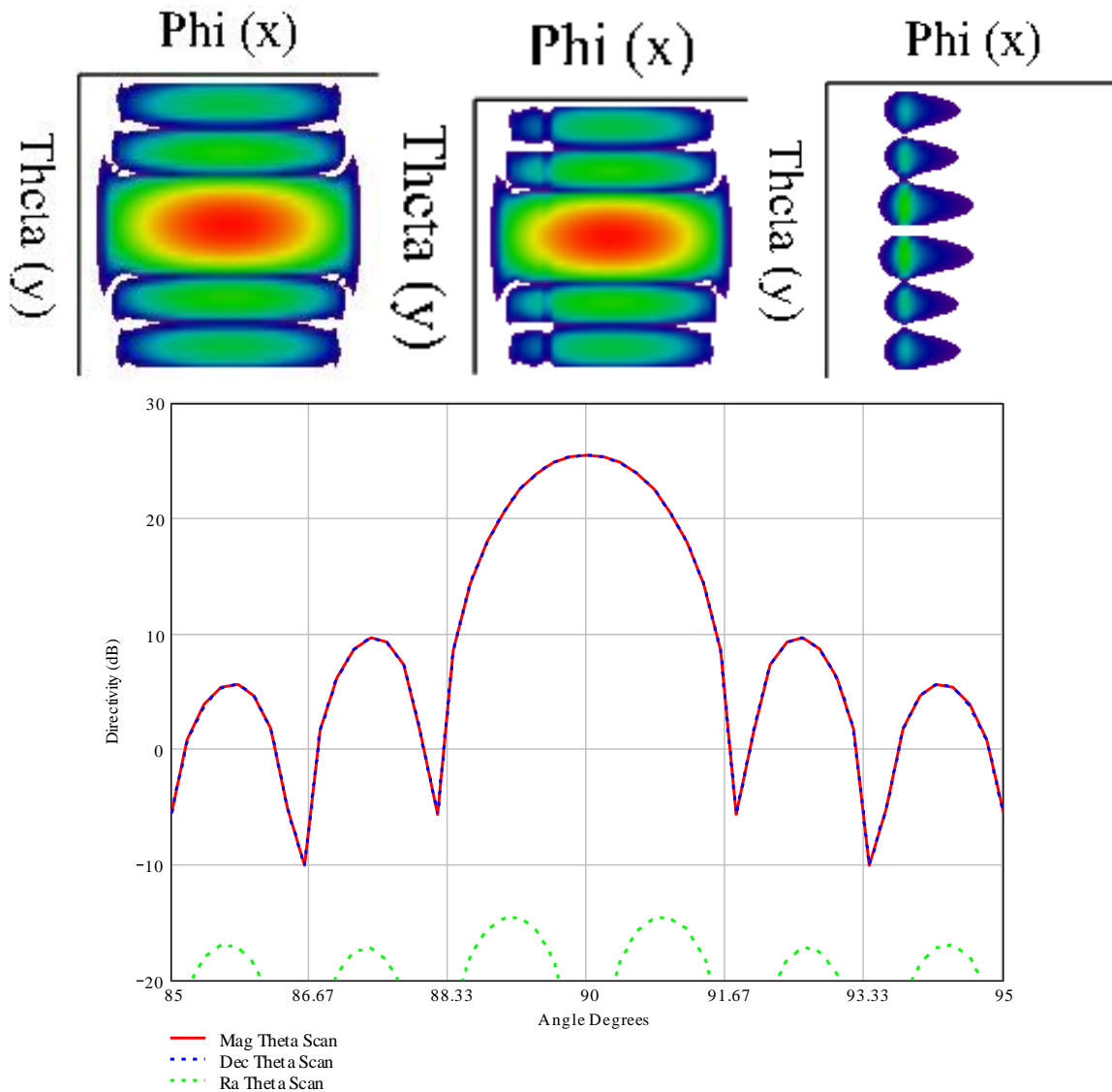


Figure 7. Simulation of a 10 meter wide x 30 meter long cylinder with a focus of 2.5 meters at 1 GHz. The feed is a single dipole oriented in the x direction that is  $\lambda/2$  long. The latitude of the telescope is 45 degrees. The color contour plots have 40 dB of range. The left plot is the magnitude, the middle plot is the declination polarization and the left plot is the right ascension polarization. The graph is a  $\theta$  scan for  $\phi=0$  degrees

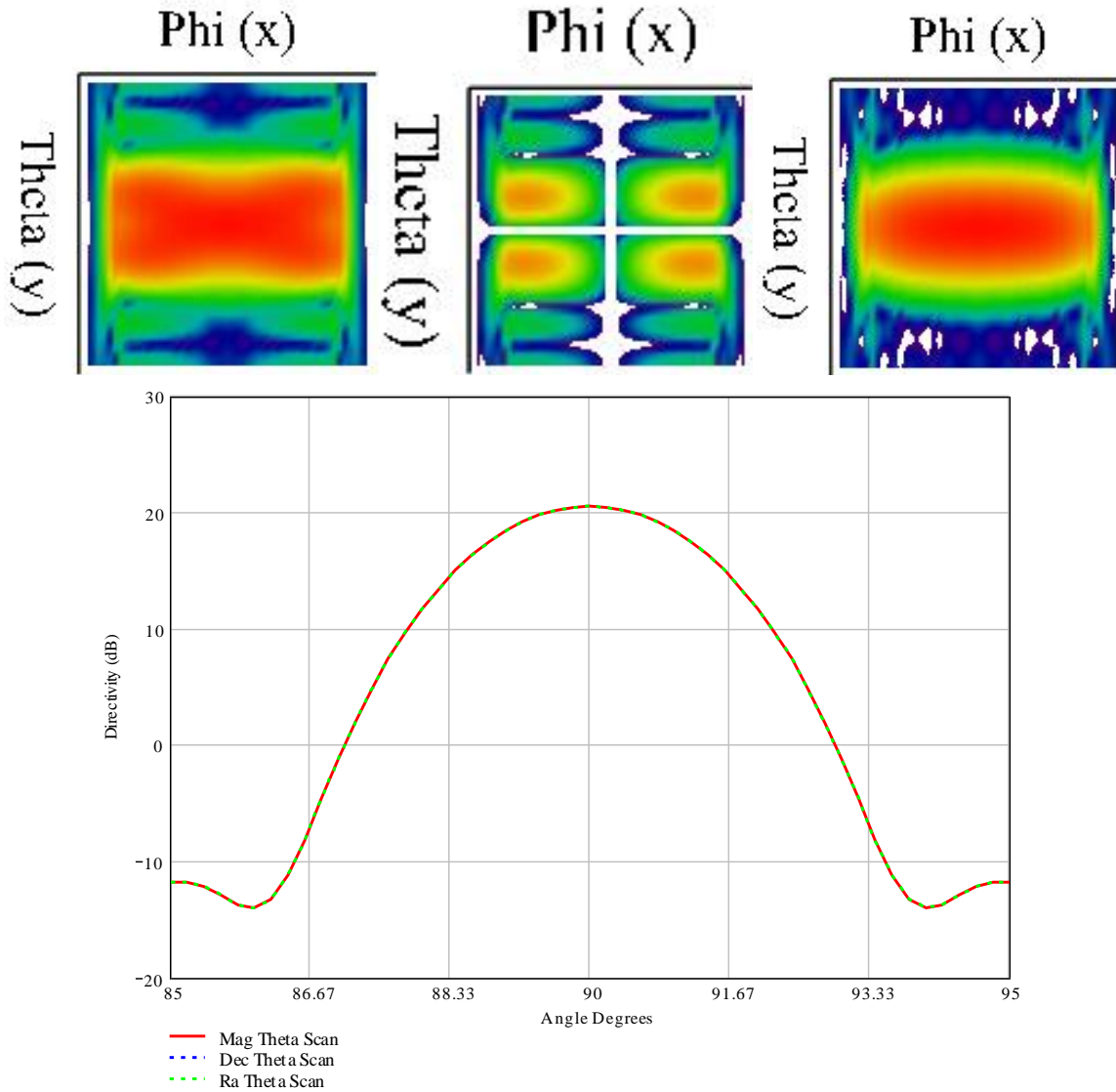


Figure 8. Simulation of a 10 meter wide x 30 meter long cylinder with a focus of 2.5 meters at 1 GHz. The feed is a single dipole oriented in the y direction that is  $\lambda/2$  long. The latitude of the telescope is 0 degrees. The color contour plots have 40 dB of range. The left plot is the magnitude, the middle plot is the declination polarization and the left plot is the right ascension polarization. The graph is a  $\theta$  scan for  $\phi=0$  degrees



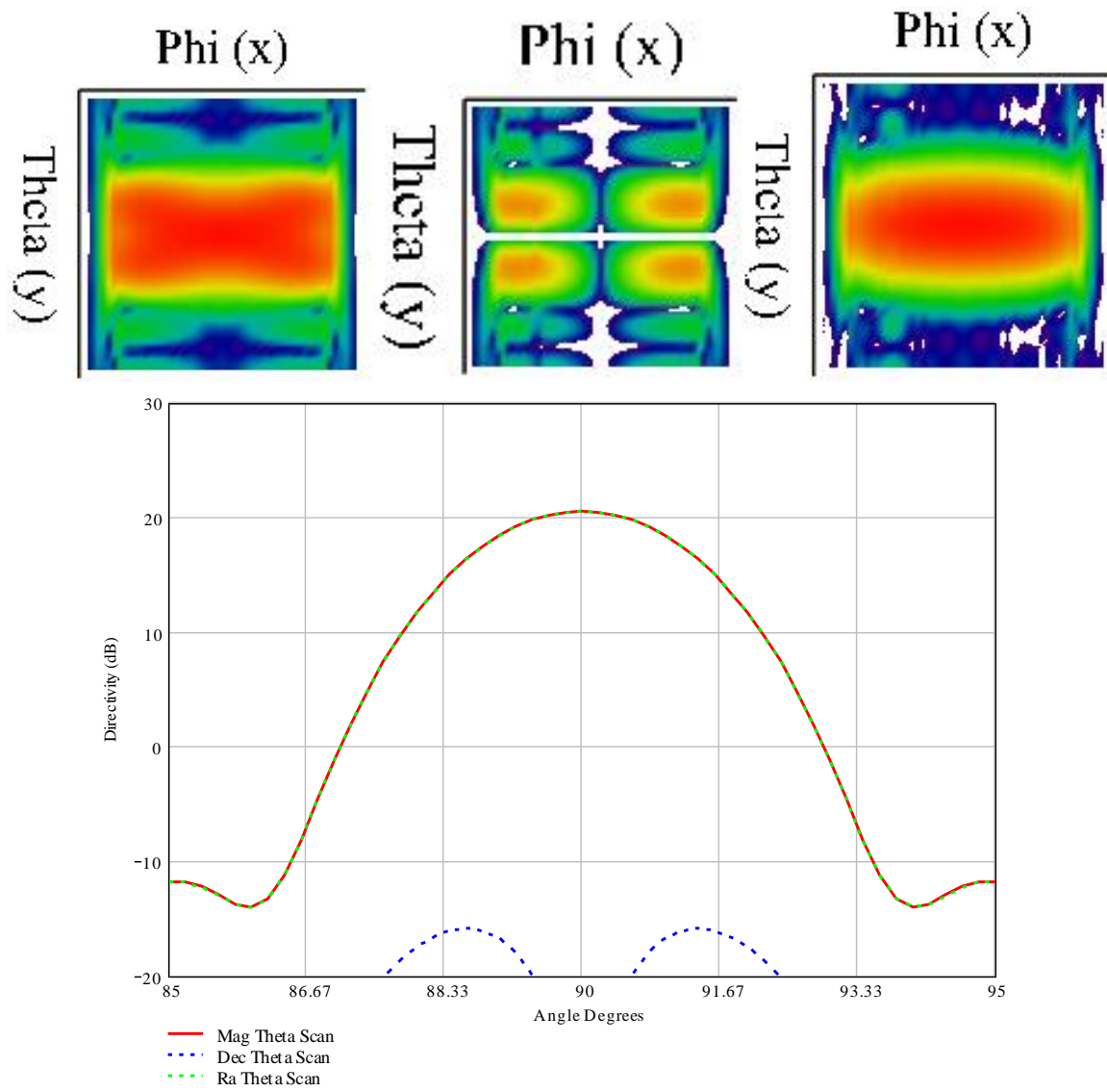


Figure 9. Simulation of a 10 meter wide x 30 meter long cylinder with a focus of 2.5 meters at 1 GHz. The feed is a single dipole oriented in the y direction that is  $\lambda/2$  long. The latitude of the telescope is 45 degrees. The color contour plots have 40 dB of range. The left plot is the magnitude, the middle plot is the declination polarization and the left plot is the right ascension polarization. The graph is a  $\theta$  scan for  $\phi=0$  degrees