Spherical Coordinates for a Parabolic Cylinder Antenna

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Introduction

The 21 cm project is based on a parabolic cylinder antenna with receivers placed periodically along the focus line of the antenna. This note proposes a set of coordinate systems to use when calculating the polarization pattern of the antenna. This coordinate system matches the system used in the note "21cm Telescope Simulation" by J. Marriner and C. Stoughton. The standard XYZ coordinate system that is presently in use for describing the antenna is shown in Figure 1. The telescope is aligned along the x-axis with the focus line of the antenna coinciding with the x axis. The parabola points up along the Z axis. For describing the antenna pattern, spherical coordinates are used. Typical spherical coordinate system have a polar angle referenced to the Z axis and an azimuthal angle for describing projections onto the XY plane as shown in *Figure 2*.

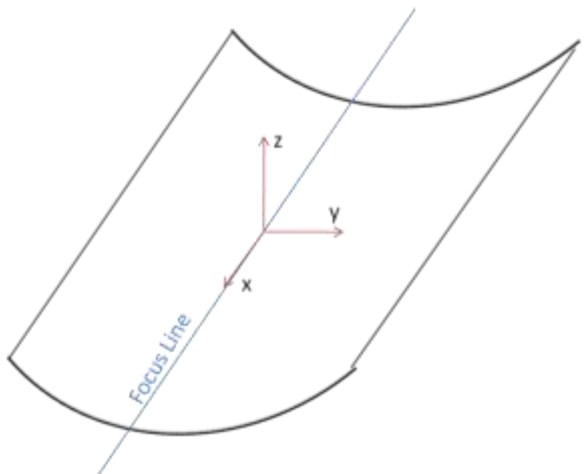

Figure 1. Coordinate system of the antenna cylinder. North is in the x direction. West is in the y direction.

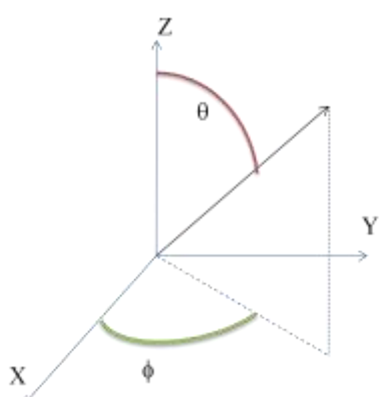

Figure 2. Spherical coordinate system referenced to the Z axis.

Directivity

The directivity of an antenna is defined as the ratio of the radiation intensity to average radiation intensity:

$$D(\phi, \theta) = \frac{U(\phi, \theta)}{\frac{1}{4\pi} \iint U(\phi, \theta) \sin(\theta) d\theta d\phi} \tag{1}$$

where the radiation intensity is given as::

$$U(\phi, \theta) = \frac{1}{2} \mathrm{Re}\{\vec{E} \times \vec{H}^*\} \cdot \hat{r} R^2 \tag{2}$$

For a parabolic cylinder antenna, the radiation pattern varies rapidly through the X-Z plane. The element angular surface area, $\sin(\theta)d\theta d\phi$, also varies rapidly near the Z axis because of the $\sin(\theta)$ dependence. This makes defining a mesh for the numerical evaluation of the integral in Equation (1) difficult. The mesh is easier to define if the polar angle, θ_T, is referenced to the Y-Z plane, and the azimuthal angle , ϕ_T, is referenced to projections in the Y-Z plane as shown in Figure 3.

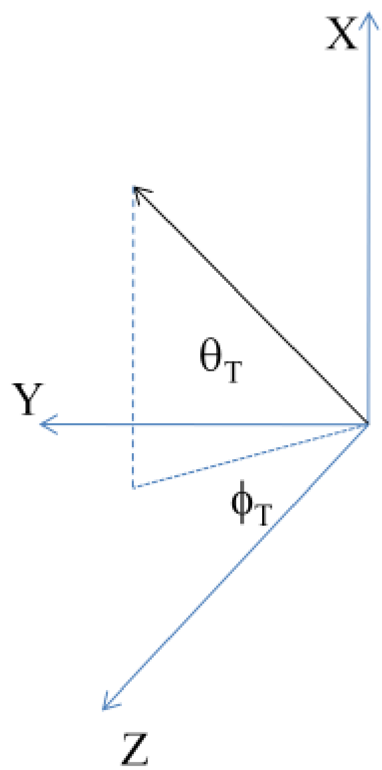

Figure 3. Spherical coordinate system referenced to the YZ plane

The integral for the directivity becomes:

$$D(\phi_T, \theta_T) = \frac{U(\phi_T, \theta_T)}{\frac{1}{4\pi} \iint U(\phi_T, \theta_T) \cos(\theta_T) d\theta_T d\phi_T} \tag{3}$$

The unit vectors for the spherical coordinate system referenced to the Y-Z plane as shown in Figure 3 are given as:

$$\begin{bmatrix}\hat{r}\\ \hat{\theta}\\ \hat{\phi}\end{bmatrix} = \begin{bmatrix}\sin(\theta_T) & \cos(\theta_T)\sin(\phi_T) & \cos(\theta_T)\cos(\phi_T)\\ \cos(\theta_T) & -\sin(\theta_T)\sin(\phi_T) & -\sin(\theta_T)\cos(\phi_T)\\ 0 & \cos(\phi_T) & -\sin(\phi)\end{bmatrix}\begin{bmatrix}\hat{x}\\ \hat{y}\\ \hat{z}\end{bmatrix} \quad (4)$$

Polarization

Because the 21 cm telescope will be used in the drift scan mode, it makes more sense to define the polarization with respect to the celestial coordinate system. For the drift scan mode, the telescope will be oriented along a north-south line. The altitude of the celestial pole (X_p axis) will be equal to the latitude of the telescope location as shown in Figure 4. The transformation between the telescope coordinate system and the celestial coordinate system is given as:

$$\begin{bmatrix}x_p\\ y_p\\ z_p\end{bmatrix} = \begin{bmatrix}\cos(\alpha) & 0 & \sin(\alpha)\\ 0 & 1 & 0\\ -\sin(\alpha) & 0 & \cos(\alpha)\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} \quad (5)$$

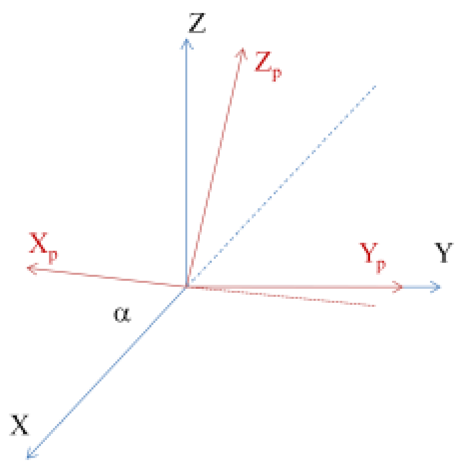

Figure 4. Celestial coordinate system (in red) with respect to telescope coordinate system. The angle α equal to the latitude of the telescope location

The celestial coordinate system is defined as by a declination angle, θ_d, and a right ascension angle, ϕ_r, as shown in Figure 5.

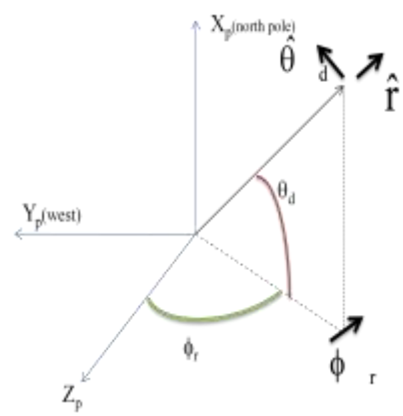

Figure 5. Celestial Coordinate system defined by declination angle, θ_d, and right ascension angle, ϕ_r.

The declination and right ascension unit vectors are given as:

$$\begin{bmatrix}\widehat{r_p}\\ \widehat{\theta_d}\\ \widehat{\phi_r}\end{bmatrix} = \begin{bmatrix}\sin(\theta_d) & -\cos(\theta_d)\sin(\phi_r) & \cos(\theta_d)\cos(\phi_r)\\ \cos(\theta_d) & \sin(\theta_d)\sin(\phi_r) & -\sin(\theta_d)\cos(\phi_r)\\ 0 & -\cos(\phi_r) & -\sin(\phi_r)\end{bmatrix}\begin{bmatrix}\widehat{x_p}\\ \widehat{y_p}\\ \widehat{z_p}\end{bmatrix} \tag{6}$$

The inverse of Equation 6 is:

$$\begin{bmatrix}\widehat{x_p}\\ \widehat{y_p}\\ \widehat{z_p}\end{bmatrix} = \begin{bmatrix}\sin(\theta_d) & \cos(\theta_d) & 0\\ -\cos(\theta_d)\sin(\phi_r) & \sin(\theta_d)\sin(\phi_r) & -\cos(\phi_r)\\ \cos(\theta_d)\cos(\phi_r) & -\sin(\theta_d)\cos(\phi_r) & -\sin(\phi_r)\end{bmatrix}\begin{bmatrix}\widehat{r_p}\\ \widehat{\theta_d}\\ \widehat{\phi_r}\end{bmatrix} \tag{7}$$

In turn, the relationship between the Cartesian unit vectors in the celestial and telescope coordinate systems is given by the inverse of Equation 5:

$$\begin{bmatrix}\hat{x}\\ \hat{y}\\ \hat{z}\end{bmatrix} = \begin{bmatrix}\cos(\alpha) & 0 & -\sin(\alpha)\\ 0 & 1 & 0\\ \sin(\alpha) & 0 & \cos(\alpha)\end{bmatrix}\begin{bmatrix}\widehat{x_p}\\ \widehat{y_p}\\ \widehat{z_p}\end{bmatrix} \tag{8}$$

Combining Equations 7 and 8:

$$\begin{bmatrix}\hat{x}\\ \hat{y}\\ \hat{z}\end{bmatrix} = \begin{bmatrix}\cos(\alpha) & 0 & -\sin(\alpha)\\ 0 & 1 & 0\\ \sin(\alpha) & 0 & \cos(\alpha)\end{bmatrix}\begin{bmatrix}\sin(\theta_d) & \cos(\theta_d) & 0\\ -\cos(\theta_d)\sin(\phi_r) & \sin(\theta_d)\sin(\phi_r) & -\cos(\phi_r)\\ \cos(\theta_d)\cos(\phi_r) & -\sin(\theta_d)\cos(\phi_r) & -\sin(\phi_r)\end{bmatrix}\begin{bmatrix}\widehat{r_p}\\ \widehat{\theta_d}\\ \widehat{\phi_r}\end{bmatrix} \tag{9}$$

The polarization unit vector for the declination angle is:

$$\widehat{\theta_d} = \begin{array}{c}\big(\cos(\alpha)\cos(\theta_d) + \sin(\alpha)\sin(\theta_d)\cos(\phi_r)\big)\hat{x}\\ +\big(\sin(\theta_d)\sin(\phi_r)\big)\hat{y}\\ +\big(\sin(\alpha)\cos(\theta_d) - \cos(\alpha)\sin(\theta_d)\cos(\phi_r)\big)\hat{z}\end{array} \tag{10}$$

The polarization unit vector for the right ascension angle is:

$$\widehat{\phi_r} = \begin{array}{c}\big(\sin(\alpha)\sin(\phi_r)\big)\hat{x}\\ -\big(\cos(\phi_r)\big)\hat{y}\\ -\big(\cos(\alpha)\sin(\phi_r)\big)\hat{z}\end{array} \tag{11}$$

The cosine and sine of the declination and right ascension angles are related to the telescope based angles by:

$$\cos(\theta_d) = \frac{\sqrt{{y_p}^2 + {z_p}^2}}{\sqrt{{x_p}^2 + {y_p}^2 + {z_p}^2}} \tag{12}$$

$$\sin(\theta_d) = \frac{x_p}{\sqrt{{x_p}^2 + {y_p}^2 + {z_p}^2}} \tag{13}$$

$$\cos(\phi_r) = \frac{z_p}{\sqrt{{x_p}^2 + {y_p}^2}} \tag{14}$$

$$\sin(\phi_r) = \frac{-y_p}{\sqrt{{x_p}^2 + {y_p}^2}} \tag{15}$$

Using Equations 4 and 5:

$$\begin{bmatrix} x_p \\ y_p \\ z_p \end{bmatrix} = \begin{bmatrix} \cos(\alpha)\sin(\theta_T) + \sin(\alpha)\cos(\theta_T)\cos(\phi_T) \\ \cos(\theta_T)\sin(\phi_T) \\ -\sin(\alpha)\sin(\theta_T) + \cos(\alpha)\cos(\theta_T)\cos(\phi_T) \end{bmatrix} \tag{16}$$