Integration Length for 21cm

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Received Power

From John Marriner's note "Digitized Response Function of a Phased Array of Antennae", the intensity of the radiation is related to the temperature. If there are many sources of radiation in the neighborhood of sky coordinates the intensities add linearly and the radiation can be characterized in terms of an intensity per steradian on the sky as follows:

$$I = 2\frac{k_B T_s}{\lambda^2} \tag{1}$$

The intensity I is measured in Watts/m^2/Hz/steradian. This can be taken as the definition of the temperature, but a blackbody radiator at temperature T_s would generate this flux per steradian per meter squared as seen from the receiving antenna. The intensity described in Equation 1 is for the sum of both modes of polarization. If the antenna is sensitive to only a single polarization mode, then the intensity available to the antenna:

$$I_{pol} = \frac{k_B T_s}{\lambda^2} \tag{2}$$

If the antenna has a solid beam angle of Ω_a steridians, then the average power per area per bandwidth is:

$$\langle s_{av} \rangle = \frac{k_B T_s}{\lambda^2}\Omega_a \tag{3}$$

The average amount of power per bandwidth received is proportional to the antenna area:

$$\langle p_{av} \rangle = \frac{k_B T_s}{\lambda^2}\Omega_a g A_{em} \tag{4}$$

where A_{em} is the maximum available area of the antenna and g is the efficiency of the antenna (g<1). The antenna beam solid angle and maximum effective area are related:

$$\lambda^2 = A_{em}\Omega_a \tag{5}$$

The average amount of power per bandwidth received by the antenna is:

$$\langle p_{av} \rangle = g k_B T_s \tag{6}$$

Assume that the antenna signal is sampled for τ_m seconds at a rate of r_s samples per second. The number of samples is:

$$N_s = \tau_m r_s \tag{7}$$

The frequency resolution of the measurement is:

$$\Delta f = \frac{1}{\tau_m} \tag{8}$$

The average power received in a bandwidth of Δf is:

$$\langle P \rangle = g k_B T_s \Delta f \tag{9}$$

If the noise temperature of the antenna receiver is T_a, then the average total power measured is:

$$\langle P_T \rangle = k_B(gT_s + T_a)\Delta f \tag{10}$$

Power Fluctuations

The amount of energy absorbed in the resolution bandwidth filter of the detector as a function of time is random. However, the average rate at which the energy is absorbed is constant. From these two preceding statements, the amount of energy absorbed in the detector in a measurement interval can be described by Poisson statistics where the mean and the standard deviation of the distribution are equal.

$$\langle \Delta {P_T}^2 \rangle = \langle P_T \rangle^2 \tag{11}$$

For many measurements of the received power, the variance of the sample mean is given by the central limit theorem

$$\langle \Delta {P_T}^2 \rangle_M = \frac{\langle \Delta {P_T}^2 \rangle}{M} \tag{12}$$

where M is the number of measurements.

Signal to Noise

Once the power spectrum has been measured, assume that the average power has been removed from the data:

$$P_{ac}(f) = P_T(f) - \langle P_T \rangle \tag{13}$$

Fluctuations in the power spectrum over a bandwidth ΔF can be expressed by the Fourier components (in k space):

$$\tilde{P}_k = \frac{1}{N} \sum_{n=-N/2}^{N/2} P_{ac}(n\Delta f + f_o) e^{-j2\pi k \frac{n}{N}} \tag{14}$$

Where Δf is the resolution bandwidth given by Equation 8 and f_o is the center frequency of the band ΔF. The number of points in the Fourier transform is given by the ratio of the bandwidth ΔF to the resolution bandwidth Δf.

$$N = \frac{\Delta F}{\Delta f} \tag{15}$$

The inverse Fourier transform is given as:

$$P_{ac}(n\Delta f + f_o) = \sum_{k=-N/2}^{N/2} \tilde{P}_k e^{j2\pi k \frac{n}{N}} \tag{16}$$

Since there is no DC component, the variance of the power spectrum becomes

$$\langle \Delta {P_{ac}}^2 \rangle = \frac{1}{N} \sum_{n=-N/2}^{N/2} \left(P_{ac}(n\Delta f + f_o) \right)^2 \tag{17}$$

From Equation 16, this can be written as:

$$\langle \Delta {P_{ac}}^2 \rangle = \frac{1}{N} \sum_{n=-N/2}^{N/2} \sum_{k=-N/2}^{N/2} \sum_{l=-N/2}^{N/2} \tilde{P}_k \tilde{P}_{-l} e^{j2\pi(k-l)\frac{n}{N}} \tag{18}$$

Since P_{ac} is real:

$$\tilde{P}_{-l} = \tilde{P}_l^{\ *} \tag{19}$$

Equation 18 can be re-written as:

$$\langle \Delta P_{ac}{}^2 \rangle = \frac{1}{N} \sum_{k=-N/2}^{N/2} \sum_{l=-N/2}^{N/2} \tilde{P}_k \tilde{P}_l^{\ *} \sum_{n=-N/2}^{N/2} e^{j2\pi(k-l)\frac{n}{N}} \tag{20}$$

Since:

$$\sum_{n=-N/2}^{N/2} e^{j2\pi(k-l)\frac{n}{N}} = N\delta_{k,l} \tag{21}$$

Equation 20 becomes

$$\langle \Delta P_{ac}{}^2 \rangle = \sum_{k=-N/2}^{N/2} \left|\tilde{P}_k\right|^2 \tag{22}$$

If the power spectrum is purely noise, then:

$$\langle \left|\tilde{P}_k\right|^2 \rangle = \frac{1}{N} \langle \Delta P_{ac}{}^2 \rangle \tag{23}$$

The variance in the P_{ac} is given by Equation 12:

$$\langle \Delta P_{ac}{}^2 \rangle = \frac{1}{M} (k_B(gT_s + T_a)\Delta f)^2 \tag{24}$$

Now assume that in addition, the power spectrum is modulated across the bandwidth ΔF with a BAO signal given as:

$$P_s(f) = P_{so} cos(2\pi k_o(f - f_o) + \phi) \tag{25}$$

The Fourier spectrum (in k space) of this signal contains only one coefficient:

$$\left|\tilde{P}_{k_o}\right|^2 = \left(\frac{P_{so}}{2}\right)^2 \tag{26}$$

The signal to noise is defined as:

$$SN = \frac{\left|\tilde{P}_{k_o}\right|^2}{\langle \left|\tilde{P}_k\right|^2 \rangle} = \frac{\Delta F}{\Delta f} \frac{1}{\langle \Delta P_{ac}{}^2 \rangle} \left(\frac{P_{so}}{2}\right)^2 \tag{27}$$

Assume that the BAO signal is some fraction, **a**, of the sky signal:

$$P_{so} = agk_B T_s \Delta f \tag{28}$$

Then the minimum number of measurements needed to obtain to resolve the modulation with a given signal to noise is:

$$M > \frac{SN}{\Delta F / \Delta f} \left(\frac{2}{a}\left(1 + \frac{1}{g}\frac{T_a}{T_s}\right)\right)^2 \tag{29}$$

The total length of making M measurements is

$$\tau_{\mathrm{t}} = M\tau_{\mathrm{m}} = \frac{SN}{\Delta F} \left(\frac{2}{a}\left(1 + \frac{1}{g}\frac{T_a}{T_s}\right)\right)^2 \tag{30}$$

Some Numbers

Antenna efficiency	g	=	0.5
Amplifier noise temperature	T_a	=	50 K
Sky Temperature	T_s	=	10 K
Sampling Rate	r_s	=	1.4 GHz
Resolution bandwidth	Δf	=	21.3 kHz
Single Measurement time	τ_m	=	46.8 uS
FFT Length	N_s	=	2^{16} = 65,536 samples
BAO signal strength	a	=	0.0001
Max. No. of BAO oscillations	N/2	=	250
Minimum BAO bandwidth	ΔF	=	10.65 MHz
Signal to Noise	SN	=	100
Number of measurements	M	=	$9.7\text{x}10^9$
Total length of integration time	τ_t	=	125.9 Hours