Directivity of a Parabolic Cylinder Antenna

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Introduction

The 21 cm project is based on a parabolic cylinder antenna with receivers placed periodically along the focus line of the antenna. This note calculates the directivity of a parabolic cylinder antenna fed to a single feed point on focus line of the antenna. The note uses the physical optics approach. Even though the 21 cm antenna is a receiver, this note will calculate the directivity of the antenna using the antenna as a transmitter. The far field in the transmitter mode is easier to calculate and the directivity of the antenna in the receiver mode is equal to the directivity in the transmit mode because of Lorentz reciprocity.

Physical Optics Approximation

In the transmitter case, the dish scatters the incident field from the antenna feed. The incident fields from the antenna feed, which is modeled as an infinitesimally short dipole, are calculated in the absence of the antenna dish. The antenna dish reflects a scattered field so that the total tangential electric field on the dish is zero. The total electric field is:

$$\vec{E}_{Total} = \vec{\mathrm{E}}_{\mathrm{inc}} + \vec{\mathrm{E}}_{\mathrm{scat}} \tag{1}$$

On the surface of the dish:

$$\hat{\mathrm{n}} \times \vec{\mathrm{E}}_{\mathrm{inc}} = -\hat{\mathrm{n}} \times \vec{\mathrm{E}}_{\mathrm{scat}} \tag{2}$$

Where n is the unit normal vector of the dish surface. Equation 2 states that if the incident field on the dish surface is known, then the scattered field on the dish surface is also known.

Equivalence Principle

The equivalence principle states the fields resulting from sources inside a closed boundary can be calculated from equivalent sources placed on the outside edge of the boundary as shown in Figure 1.

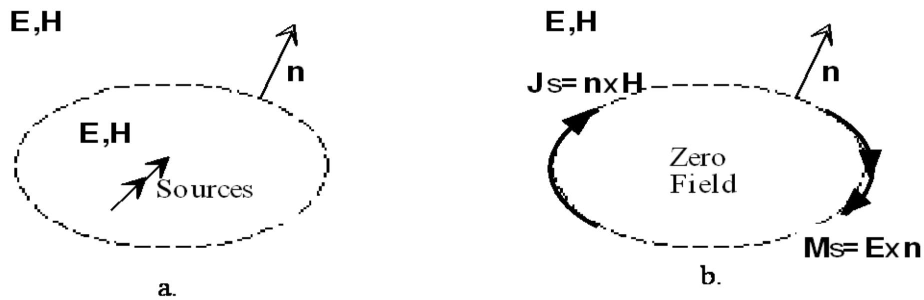

Figure 1. Illustration of the equivalence principle. The fields shown in (a) and (b) are the same.

Since the fields are zero just inside the boundary any material can be placed inside the boundary and not affect the fields outside of the boundary. In Figure 2a, the boundary is replaced by an electric conductor ($E_{tan} = 0$) which shorts out the electric current J_S. Likewise, in Figure 2b, the boundary is replaced by a magnetic conductor ($H_{tan} = 0$) which shorts out the magnetic current M_S.

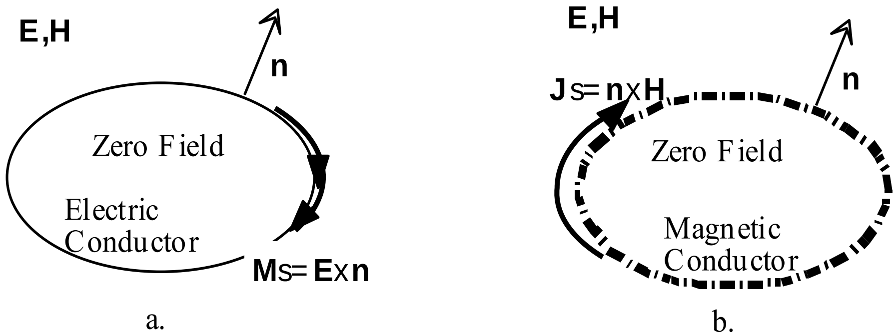

Figure 2. Equivalence principle with electric and magnetic conductors

The geometry of the parabolic cylinder is shown in Figure 3. Using the equivalence principle, the scattered field is produced by a magnetic surface current residing on the metal surface. Using, the method of images, the metal surface can be removed if the magnetic surface current is increased by a factor of two.

Figure 3. On the left is the equivalent magnetic surface current that produces the scattered field from the dish. On the right, is the equivalent magnetic surface current using the method of images and the dish removed.

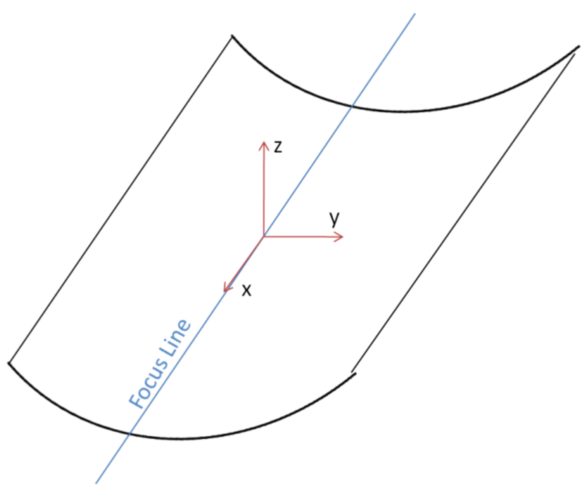

Figure 4. Coordinate system of the antenna cylinder

The magnetic surface current with the dish removed is given as:

$$\overrightarrow{\mathrm{M}}_s = 2\hat{\mathrm{n}} \times \mathrm{E}_{\mathrm{inc}} \tag{3}$$

Antenna Geometry

In this note, the length of the antenna is oriented along the x axis and the width is oriented along the y axis as shown in Figure 4. The focus line runs though the origin along the x axis. The feed of the antenna is located on the focus line at the origin. The feed is modeled as infinitesimally short dipole oriented along the x axis and another infinitesimally short dipole oriented along the y axis so that both polarizations can be modeled. The parabolic shape of the antenna is given as:

$$z'(y') = f\left[\left(\frac{y'}{2f}\right)^2 - 1\right] \tag{4}$$

where f is the focal length of the parabola. The unit vector normal to the antenna surface is given as:

$$\hat{n}(y') = \frac{1}{\sqrt{y'^2 + 4f^2}}\begin{pmatrix} 0 \\ -y' \\ 2f \end{pmatrix} \tag{5}$$

Incident Electric Field

The antenna feed is modeled as infinitely short dipole. An infinitely short dipole has a current density of the form:

$$\vec{\mathrm{J}} = \overrightarrow{\mathrm{I}\Delta\mathrm{l}}\,\delta(\mathrm{x} - \mathrm{x}')\delta(\mathrm{y} - \mathrm{y}')\delta(\mathrm{z} - \mathrm{z}') \tag{6}$$

The magnetic vector potential is determined from the following equation:

$$\nabla^2\vec{A} + \beta^2\vec{A} = -\vec{J} \tag{7}$$

where:

$$\beta = \omega\sqrt{\mu\varepsilon} \tag{8}$$

The solution for the magnetic vector potential for an infinitely short dipole is:

$$\vec{\mathrm{A}} = \overrightarrow{\mathrm{I}\Delta\mathrm{l}}\,\frac{\mathrm{e}^{-\mathrm{j}\beta|\vec{\mathrm{r}}-\vec{\mathrm{r}}'|}}{4\pi|\vec{\mathrm{r}} - \vec{\mathrm{r}}'|} \tag{9}$$

The magnetic field is the derivative of the magnetic vector potential:

$$\vec{H} = \vec{\nabla} \times \vec{A} \tag{10}$$

The electric field in a source free region is determined from the curl of the magnetic field

$$\vec{E} = \frac{1}{j\omega\epsilon}\vec{\nabla} \times \vec{H} \tag{11}$$

The free space permittivity and permeability determine the wave impedance and phase velocity in free space:

$$v_{ph} = \frac{\omega}{\beta} = \frac{1}{\sqrt{\mu\epsilon}} \tag{12}$$

and

$$\eta = \sqrt{\frac{\mu}{\epsilon}} \tag{13}$$

The electric field is found from the magnetic vector potential

$$\vec{E} = -j\frac{\eta}{\beta}\vec{\nabla} \times \vec{\nabla} \times \vec{A} \tag{14}$$

Expanding Equation 14:

$$\begin{aligned}\vec{E}(\vec{r}) = -j\frac{\eta}{\beta}\iiint \frac{\mathrm{e}^{-\mathrm{j}\beta\mathrm{R}}}{4\pi\mathrm{R}^3}\frac{3(1+\mathrm{j}\beta\mathrm{R}) - (\beta\mathrm{R})^2}{\mathrm{R}^2}(\vec{\mathrm{r}} - \vec{\mathrm{r}}') \\ \times \left((\vec{\mathrm{r}} - \vec{\mathrm{r}}') \times \vec{\mathrm{J}}(\vec{\mathrm{r}}')\right)\mathrm{d}^3\mathrm{r}' \\ - j\frac{\eta}{\beta}\iiint \frac{\mathrm{e}^{-\mathrm{j}\beta\mathrm{R}}}{4\pi\mathrm{R}^3}2(1+\mathrm{j}\beta\mathrm{R})\vec{\mathrm{J}}(\vec{\mathrm{r}}')\mathrm{d}^3\mathrm{r}'\end{aligned} \tag{15}$$

where

$$R = |\vec{\mathrm{r}} - \vec{\mathrm{r}}'| \tag{16}$$

and

$$R^2 = (x - x')^2 + (y - y')^2 + (z - z')^2 \tag{17}$$

Scattered Electric field

For magnetic currents, which are the dual of electric currents, Maxwell's equations become:

$$\vec{\nabla} \times \vec{E} = -j\omega\mu\vec{H} - \vec{M} \tag{18}$$

and

$$\vec{\nabla} \times \vec{H} = j\omega\epsilon\vec{E} \tag{19}$$

Analogously, the electric field can be found from an electric vector potential:

$$\vec{E} = -\vec{\nabla} \times \vec{F} \tag{20}$$

where the electric vector potential is determined from the magnetic current density

$$\nabla^2\vec{F} + \beta^2\vec{F} = -\vec{M} \tag{21}$$

For an infinitesimal magnetic surface current over an area ΔA, the green's function for the electric vector potential is:

$$\vec{\mathrm{F}} = \vec{\mathrm{M}}_s\Delta\mathrm{A}\frac{\mathrm{e}^{-\mathrm{j}\beta|\vec{\mathrm{r}} - \vec{\mathrm{r}}'|}}{4\pi|\vec{\mathrm{r}} - \vec{\mathrm{r}}'|} \tag{22}$$

The electric field due to the distribution of magnetic surface current is:

$$\vec{E}(\vec{r}) = \iint \frac{\mathrm{e}^{-\mathrm{j}\beta\mathrm{R}}}{4\pi\mathrm{R}^3}(1+\mathrm{j}\beta\mathrm{R})\left((\vec{\mathrm{r}} - \vec{\mathrm{r}}') \times \vec{\mathrm{M}}_s(\vec{\mathrm{r}}')\right)\mathrm{dA}' \tag{23}$$

Using Equation (3), the scattered field is:

$$\vec{E}_{scat}(\vec{r}) = \iint \frac{e^{-j\beta R}}{4\pi R^3}(1 + j\beta R)\left((\vec{r} - \vec{r}') \times \left(2\hat{n}(\vec{r}') \times E_{inc}(\vec{r}')\right)\right) dA' \tag{24}$$

where the incident field as a function of feed current is given by Equation 15.

Parallel Ray Approximation

In the transmit case, only the far fields are of interest. In the far field:

$$|\vec{r}| \gg |\vec{r}'| \tag{25}$$

and

$$e^{-j\beta R} = e^{-j\beta|\vec{r}|} e^{j\beta\hat{r}\cdot\vec{r}'} \tag{26}$$

where

$$\hat{r} = \hat{x}\cos(\phi)\sin(\theta) + \hat{y}\sin(\phi)\sin(\theta) + \hat{z}\cos(\theta) \tag{27}$$

Equation (24) becomes

$$\vec{E}_{scat}(\vec{r}) \approx j\beta \frac{e^{-j\beta R}}{4\pi R} \iint e^{j\beta\hat{r}\cdot\vec{r}'}\left(\hat{r} \times \left(2\hat{n}(\vec{r}') \times E_{inc}(\vec{r}')\right)\right) dA' \tag{28}$$

Directivity, Beam Solid Angle, and Effective Aperture

Since the far-fields fall off as 1/R, the radiation intensity is defined as:

$$U(\phi, \theta) = \frac{1}{2}\mathrm{Re}\left\{\vec{E} \times \vec{H}^*\right\} \cdot \hat{r}R^2 \tag{29}$$

The directivity is defined as the ratio of the radiation intensity to average radiation intensity:

$$D(\phi, \theta) = \frac{U(\phi, \theta)}{\frac{1}{4\pi}\iint U(\phi, \theta)\sin(\theta)d\theta d\phi} \tag{30}$$

The beam solid angle is defined as:

$$\Omega_A = \frac{4\pi}{D_{max}} \tag{31}$$

The effective aperture is defined as:

$$A_e = \frac{\lambda^2}{\Omega_A} \tag{32}$$

where λ is the wavelength.

Simulation Results

A short Java program was written to calculate the directivity of a parabolic cylinder antenna with a single feed but with two polarizations. The user specifies the frequency, length, width and focus of the antenna and the program computes the directivity as a function of polar angle. Figure 5 shows the directivity plots for both polarizations for an single feed antenna that is 30 m in length, 2 meters in width with a focus of 1 meter. Figure 6 is the same antenna but the length is 10 meters. Significant ripple shows up in the phi scan which is parallel to the antenna length for the Y polarization. Figure 7is the same antenna but with a 10 meter length and a focus of two meters.

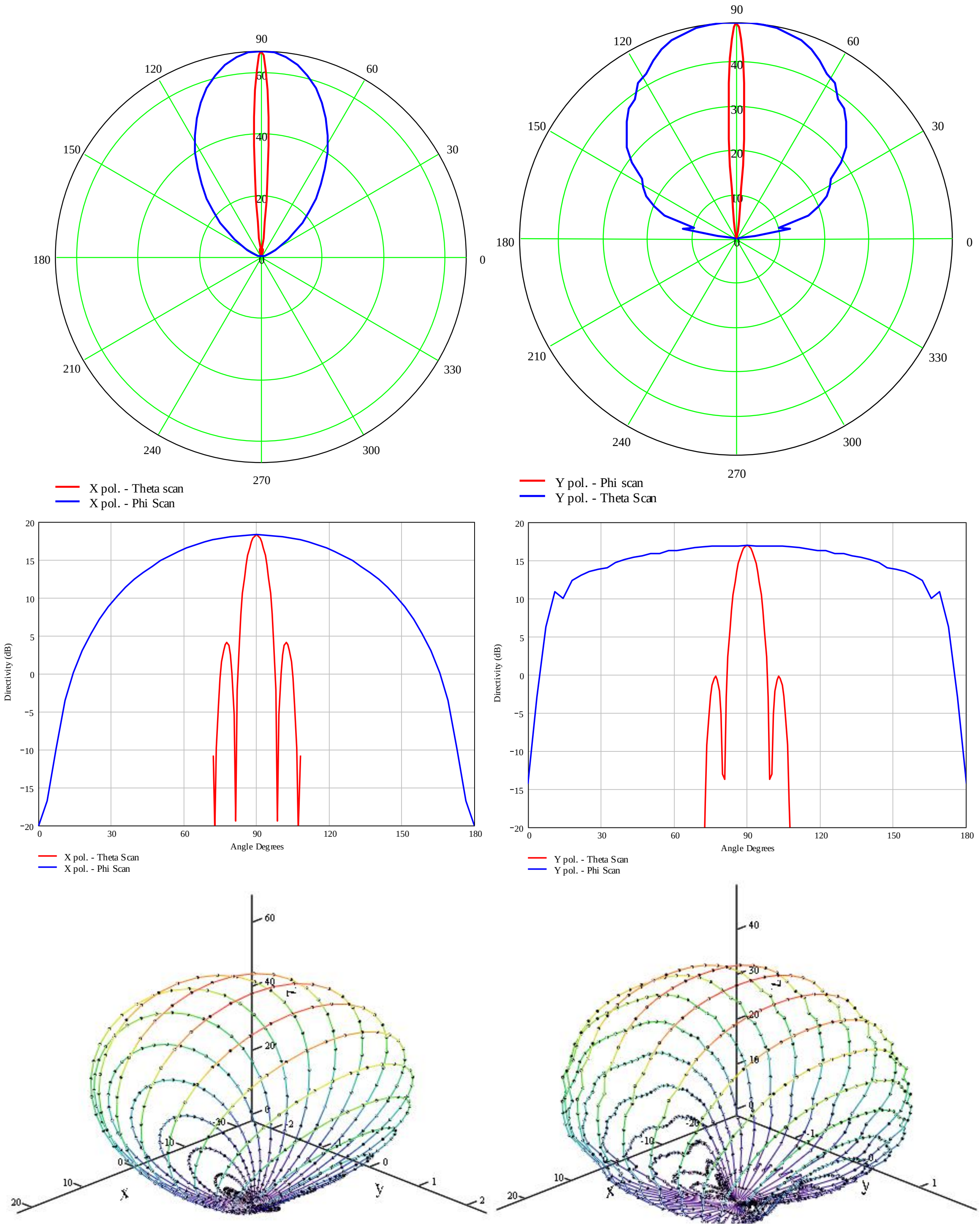

Figure 5. Theta scan (perpendicular to the antenna length)and Phi scan (parallel to the antenna length) of the x (left picture) and y (right picture) polarization for an single feed antenna that is 30 m in length, 2 meters in width with a focus of 1 meter.

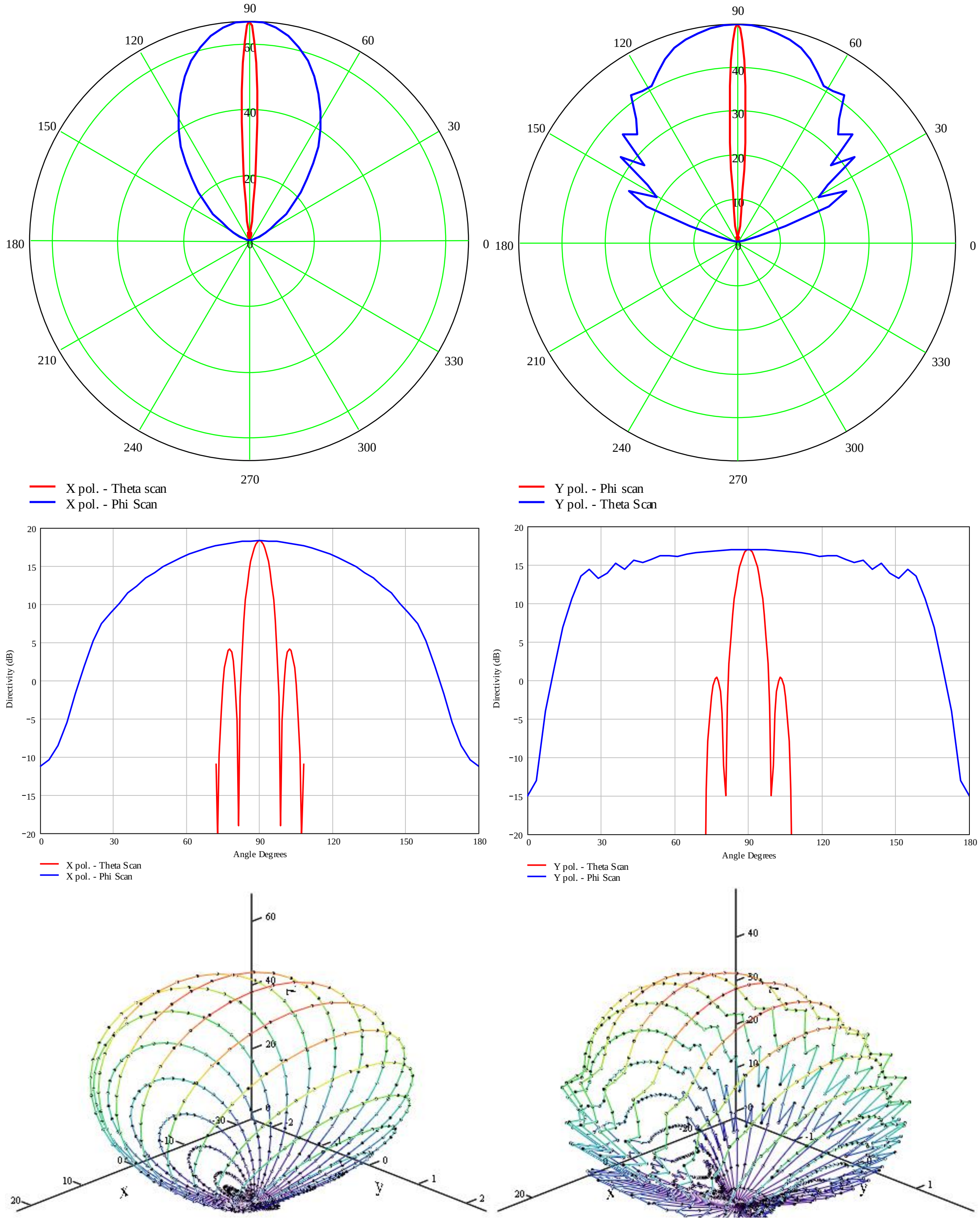

Figure 6. Theta scan (perpendicular to the antenna length) and Phi scan (parallel to the antenna length) of the x (left picture) and y (right picture) polarization for an antenna that is 10 m in length, 2 meters in width with a focus of 1 meter.

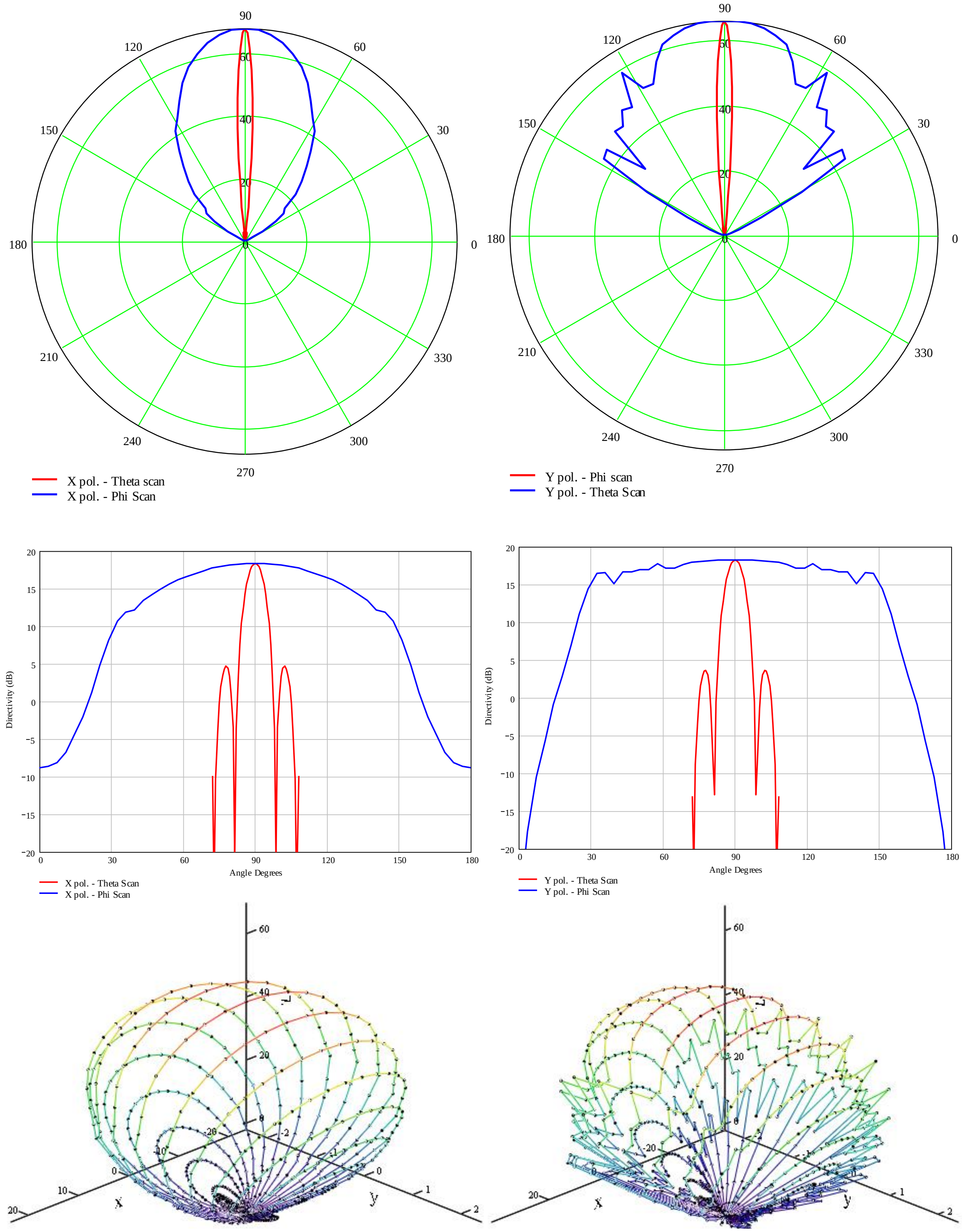

Figure 7. Theta scan (perpendicular to the antenna length) and Phi scan (parallel to the antenna length) of the x (left picture) and y (right picture) polarization for an antenna that is 10 m in length, 2 meters in width with a focus of 2 meters.